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Abstract

Method of moment estimators are generally obtained by adopting orthogonality conditions, in which particular functions in terms of the observed data and unknown parameters are supposed to have zero expectation. For regression models this implies exploiting presumed uncorrelatedness of the model disturbances and identifying instrumental variables. Here, utilizing non-orthogonality conditions is examined for linear cross-section multiple simultaneous regression models. Employing flexible bounds on the correlations between disturbances and regressors one avoids: (i) adoption of often incredible and unverifiable strictly zero correlation assumptions, and (ii) imprecise inference due to possibly weak or invalid instruments. The asymptotic validity of the suggested alternative form of inference is proved and its finite sample accuracy is demonstrated by simulation. It enables to produce inference on coefficient values that within constraints is endogeneity robust. Also a sensitivity analysis of standard least-squares or instrument-based inference is possible, and even a test of the in the standard approach unavoidable though "non-testable" exclusion restrictions regarding external instruments. The practical relevance is illustrated in a few applications borrowed from the textbook literature.

1. Introduction

The standard quasi-experimental approach in applied econometric research requires the adoption of so-called orthogonality conditions. An initial set of such conditions has to be justified on the basis of persuasive common sense or economic-theoretical arguments. However, contenders may easily disqualify these arguments as opportunistic subjective beliefs, since these conditions cannot be vindicated by empirical statistical evidence

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without the adoption of further non-testable conditions. Although the formal testing of any overidentification restrictions is feasible, its interpretation is contingent on the validity of the initial just-identifying set of non-testable orthogonality conditions. For the analysis of single regression equations this implies that at least as many excluded variables have to be proclaimed as being uncorrelated with the unobservable random disturbances of the equation as there are unknown coefficients of the endogenous explanatory variables, which are those that could be correlated with the disturbances. So these excluded variables, also addressed as the external instruments, should have no direct effect on the dependent variable of the equation of interest. Moreover, for yielding effective inference, they should at the same time have a substantial correlation with the endogenous regressors of the relationship.

Here it will be shown that there is an alternative route towards identification by adopting non-orthogonal moment conditions, which may in fact be more credible than the just-identifying orthogonality conditions, because the moments concerned do not have to be strictly zero but may vary over an interval. A simple implementation of this yields a mutation of the ordinary least-squares (OLS) estimator. It is consistent if the correlation between all regressors and the disturbance is known. The limiting distribution of this unfeasible estimator enables to construct feasible asymptotic test procedures for regression coefficients. These exploit, in addition to some standard orthogonality conditions with respect to any exogenous regressors, also bounds on the degree of non-orthogonality or simultaneity of the endogenous regressors. In simulations it is demonstrated that even in quite small samples these tests have appropriate level control and impressive power. They can be converted into feasible asymptotic confidence intervals with conservative coverage, which are often more informative than intervals obtained by instrumental variables (IV) methods, especially when some instruments are weak.

Moreover, this robustified OLS-based inference enables to test exclusion restrictions in the following way: it can produce the set of values of simultaneity correlations which endorses the exclusion restrictions, as well as the set under which these are rejected at a chosen significance level. Thus, depending on the span and the location of these two sets, the credibility of exclusion restrictions can in this way be supported or wiped away by even in small samples very accurate statistical evidence.

The methods developed here are based on different assumptions than those made under the standard approach. Fewer assumptions, because no external instruments and corresponding exogeneity assumptions are required. On the other hand, some extra assumptions have to be adopted, namely on the third and fourth moments of the regressors and disturbances because these determine the variance of the adapted least-squares estimator. And, when it comes to making decisions on the basis of the produced inference, this has to be confronted with an opinion of the researcher on the likely degree of endogeneity. So, basically, strict exogeneity assumptions on external instruments are exchanged for interval assumptions with respect to the endogeneity of the regressors.

The techniques developed here are a generalization of (now allowing for an arbitrary number of endogenous and exogenous regressors) and an extension to (also enabling general tests of linear restrictions on the coefficients of both endogenous and exogenous regressors) some basic initial results already published in Kiviet (2013, Section 4) and further justified in Kiviet (2016). This approach, addressed as kinky least-squares (KLS), was triggered by some rudimentary findings dating back to Goldberger (1964, p.359) and

Rothenberg (1972). That even the standard OLS estimator often beats IV or two-stage least-squares (TSLS) on a mean squared errors criterion has already been demonstrated in Kiviet and Niemczyk (2012), Kiviet (2013) and Doko Chatoka and Dufour (2016). That the here presented generalized KLS procedures are preferable to standard OLS, and in many cases also to IV, is because they enable accurate statistical inference in models with endogenous regressors, while avoiding the hazards of weakness or invalidity of instruments.

For an overview of complicating issues undermining the accuracy of statistical inference in models with endogenous regressors due to employment of weak or invalid external instruments, see for instance Dufour (2003). In essence these complications are fourfold: (i) under weak though valid instruments standard asymptotic IV inference is inaccurate (yields seriously biased non-normal coefficient estimates with poorly assessed standard errors, resulting in bad level control of tests), (ii) employing more sophisticated weak-instrument techniques may result in improved level control, but yields confidence sets which are often very wide or even unbounded, (iii) the use of invalid instruments produces as a rule highly inaccurate inference, whereas (iv) testing the validity of particular instruments seems only possible when a sufficient number of valid instruments is already available. During the last decades (i) and (ii) received a lot of attention in the literature. The present study addresses the two more fundamental problems (iii) and (iv). It escapes from problem (iii) by developing a formal frequentist approach to produce accurate inference in simultaneous models not employing any external instrumental variables at all by incorporating into the analysis an interval assumption on the degree of simultaneity. When implemented as an exclusion restrictions test this approach also allows to break out of the vicious circle of problem (iv) through testing the validity of instruments without requiring any untested orthogonality conditions. Various other studies have addressed problem (iii). The degree of invalidity of instruments is incorporated into a frequentist analysis by Ashley (2009)[1] and by Bayesian methods in Kraay (2012). Nevo and Rosen (2012) derive set estimates under assumptions on the signs and relative magnitudes of the simultaneity and instrument invalidity. Conley et al. (2012) augment the model with the instruments and make assumptions on its coefficients (which would be zero under correct exclusion) which next allow frequentist or Bayesian methods to obtain inference allowing for instrument invalidity. However, unlike ours, all these approaches still employ IV methods and so do not escape from problem (i) nor (ii). To our knowledge feasible tests for (iv) have not been developed before, apart from various informal procedures, such as suggested in, for instance, Bound and Jaeger (2000) and Altonji et al. (2005).

In Section 2, after having reviewed how in a linear multiple regression equation with some endogenous regressors consistent estimators can be obtained by exploiting classic identifying orthogonality conditions, we demonstrate how this can also be achieved by adopting non-orthogonality conditions. This yields an adapted least-squares estimator which is a function of the nuisance parameter vector containing the correlations between all the regressors and the disturbances. The limiting distribution of this unfeasible estimator is presented in Section 3 (and derived in Appendices); from this a feasible test procedure for a set of general restrictions on the coefficient values readily follows. Section 4 demonstrates how this procedure can be employed for testing any exclusion

[1] Kiviet (2016) addresses flaws in the asymptotic derivations in offsprings of this study.

restrictions relevant within the context of a classic instrumental variables based analysis. Section 5 provides simulation results on size and power of exclusion restriction tests in simultaneous models with one or two endogenous regressors. Section 6 demonstrates how the various techniques can be employed in practice by analyzing three empirical data sets used for illustrative purposes in well-known textbooks. Section 7 concludes.

2. Two distinct approaches towards identification

Consider a sample of n independent and identically distributed observations $\{y_i, x_i'; i = 1, ..., n\}$ on a linear causal relationship given by

$$y_i = x_i'\beta + u_i, \text{ with } x_i \sim (0, \Sigma_{xx}) \text{ and } u_i \sim (0, \sigma_u^2), \tag{2.1}$$

where β is an unknown constant $K \times 1$ coefficient vector, $K \times K$ matrix Σ_{xx} is positive definite with all its elements finite and $0 < \sigma_u^2 < \infty$. Vector $x_i' = (x_i^{(1)\prime}, x_i^{(2)\prime})$ has $K_1 + K_2 = K$ elements, such that

$$E(x_i^{(1)} u_i) = \sigma_{x^{(1)}u} \text{ and } E(x_i^{(2)} u_i) = 0, \tag{2.2}$$

with $\sigma_{x^{(1)}u} \in \mathbb{R}^{K_1}$ which in practice is generally unknown.

Model (2.1) with zero mean regressors may actually originate from a model with the same disturbances, the same K slope coefficients corresponding to regressors with an arbitrary observation specific mean, and including as well an unknown intercept. Then taking all observations in deviation from their expectation annihilates the intercept and results in model (2.1) with zero-mean regressors.

A non-zero but constant correlation between elements of the $K_1 \times 1$ vector of regressors x_{1i} and the disturbance u_i may be due either to simultaneity (elements of $x_i^{(1)}$, while constituting causes for y_i, are causally dependent on y_i themselves too), or to measurement errors in $x_i^{(1)}$, or perhaps to particular omissions in the regression specification. Generally, such non-zero correlations render all elements of the OLS estimator $\hat{\beta}_{OLS} = (\Sigma_{i=1}^n x_i x_i')^{-1} \Sigma_{i=1}^n x_i y_i$ biased for β and inconsistent under common regularity conditions. A standard method to achieve consistent estimators is reviewed in the next subsection, followed in a second subsection by the development of an alternative non-standard procedure which in a sense repairs the inconsistency of least-squares.

2.1. Exploiting orthogonality conditions

The standard approach to achieve identification and consistent estimation of β is to find a $L_2 \times 1$ vector of observations $z_i^{(2)}$, such that for the $L \times 1$ vector $z_i' = (x_i^{(2)\prime}, z_i^{(2)\prime})$, with $L = K_2 + L_2 \geq K$, one is willing to assume validity of the orthogonality conditions

$$E(z_i u_i) = 0, \ \forall i. \tag{2.3}$$

These imply

$$E(z_i y_i) = E(z_i x_i')\beta, \ \forall i.$$

Next, the "analogy principle" of the method of moments, by which expectations are replaced by corresponding sample averages, suggests as an estimator for β the "best" solution $\hat{\beta}$ of $n^{-1}\Sigma_{i=1}^n z_i y_i = (n^{-1}\Sigma_{i=1}^n z_i x_i')\,\hat{\beta}$, hence of

$$Z'y = Z'X\hat{\beta},$$

where $X = (x_1, ..., x_n)'$, $y = (y_1, ..., y_n)'$ and $Z = (z_1, ..., z_n)'$. In case $L = K$ and $Z'X$ has full rank there is a unique solution, namely

$$\hat{\beta}_{IV} = (Z'X)^{-1}Z'y, \tag{2.4}$$

which realizes $Z'(y - X\hat{\beta}_{IV}) = 0$, thus achieving orthogonality of the residuals $\hat{u}_{IV} = y - X\hat{\beta}_{IV}$ and the instruments in the sample, similar to the zero moments $E(Z'u) = 0$.

If $L > K$, while $X'Z$ has rank K and $Z'Z$ has rank L, then orthogonality of all individual instruments and the residuals cannot be achieved, but a unique solution is found by minimizing a quadratic form in the vector $Z'(y - X\hat{\beta})$, namely $(Z'y - Z'X\hat{\beta})'W(Z'y - Z'X\hat{\beta})$, where W is some symmetric positive definite weighing matrix. This yields the estimator $\hat{\beta}_{WIV} = (X'ZWZ'X)^{-1}X'ZWZ'y$. Under standard regularity conditions $\hat{\beta}_{WIV}$ (like $\hat{\beta}_{IV}$) is consistent and has a limiting normal distribution. When $L > K$ the efficient Generalized Method of Moment (GMM) estimator is obtained by choosing W proportional to $[Z'Var(u)Z]^{-1}$, where $u = (u_1, ..., u_n)'$. Because we have here $Var(u) = \sigma_u^2 I$ this simplifies to the TSLS estimator

$$\hat{\beta}_{TSLS} = (X'P_ZX)^{-1}X'P_Zy, \tag{2.5}$$

where $P_Z = Z(Z'Z)^{-1}Z'$. If $L > K$ it does not realize in the sample orthogonality of Z and $\hat{u}_{TSLS} = y - X\hat{\beta}_{TSLS}$, but it does realize the orthogonality relationships $\hat{X}'\hat{u}_{TSLS} = 0$, with $\hat{X} = P_ZX$ the orthogonal projection of the K regressors X on the L dimensional sub-space spanned by the instrumental variables Z.

In the above approach, validity of all inference is based especially on validity of the orthogonality conditions (2.3). In case $L = K$ no statistical evidence can be produced from the sample under study on this validity, because then $Z'(y - X\hat{\beta}_{IV})$ equals zero by construction. Self-evidently, validity (orthogonality) of the instruments $z_i^{(2)}$ requires validity of the L_2 zero restrictions $\beta_z^{(2)} = 0$ when model (2.1) is extended with $z_i^{(2)\prime}\beta_z^{(2)}$, because $\beta_z^{(2)} \neq 0$ would jeopardize $E(z_i^{(2)}u_i) = 0$. Thus, these zero or exclusion restrictions should be valid.[2] However, they cannot be properly tested unless we would have another K_1 valid instruments in order to cope with the endogeneity of $x_i^{(1)}$. But, for these we would only be able to test their exclusion restrictions unless ..., and so on. More generally, when the model with $L \geq K$ instruments forms the starting point and K of these instruments are presupposed to be valid, then the validity of only $L - K$ exclusion restrictions, establishing $L - K$ over-identification restrictions, can be tested. This is equivalent with testing the validity of $L - K$ instruments in addition to K valid –though untested– instruments. Within the present context, testing the validity of this initial just-identifying set of K instruments is simply impossible.[3]

This impossibility is highly uncomfortable, because the interpretation of the outcome of overidentification or instrument validity tests is conditional on the legitimacy of adopting K non-testable zero correlation assumptions. This embodies the Achilles heel of many applied econometric studies. This vulnerability can only be concealed by wrapping this limb into non-statistical often highly speculative rhetoric arguments. Such a verbal cover-up often provides just meager protection against dissident views.

[2] For more on the correspondence between testing (incremental or difference) overidentifying restrictions by Sargan-Hansen tests and testing exclusion restrictions see Kiviet (2017).

[3] On the impossibility to test the exogeneity of all the instruments, see for instance Stock and Watson (2003, p.352): "Assessing whether the instruments are exogenous *necessarily* requires making an expert judgement based on personal knowledge of the application."

2.2. Exploiting some non-orthogonality conditions as well

Next consider an alternative to the above standard approach regarding achieving identification. Instead of adopting $L \geq K$ orthogonality conditions $E(z_i u_i) = 0$, which imply zero correlation between each instrument and the disturbance, consider adopting a numerical assumption concerning the K elements of the correlation vector $\rho_{xu} = (\rho_{x_1 u}, ..., \rho_{x_K u})'$, where

$$\rho_{x_j u} = E(x_{ji} u_i)/(\sigma_u \sigma_j), \ j = 1, ..., K,$$

with σ_j^2 equal to the j-th diagonal element of matrix Σ_{xx}. Hence, suppose we replace the assumption $\rho_{zu} = (\rho_{z_1 u}, ..., \rho_{z_L u})' = 0$ by

$$\rho_{xu} = r, \tag{2.6}$$

where $r = (r_1, ..., r_K)'$ with scalar r_j the adopted value of the correlation between x_{ij} and u_i, so $|r_j| < 1$, $\forall j$. This implies adopting the K moment conditions

$$E(x_{ij} u_i) = \sigma_{x_j u} = r_j \sigma_u \sigma_j, \ j = 1, ..., K. \tag{2.7}$$

If we are still convinced of the exogeneity of the regressors $x_i^{(2)}$ we could have

$$r_j = 0 \text{ for } j = K_1 + 1, ..., K \text{ and } r_j \neq 0 \text{ otherwise.}$$

Then we adopt K_2 standard orthogonality conditions and K_1 non-orthogonality conditions.

One may object that in practice one generally would not know the true values of the elements of ρ_{xu}, so r will generally differ from ρ_{xu}. Although true, this will turn out to be of moderate concern, because in the analysis to follow r_j will not necessarily be kept fixed, but may vary within the interval $(-1, +1)$. Moreover, in the classic approach the adopted strictly zero values for the L elements of ρ_{zu} may be false too, raising far more serious credibility issues, because this approach does not allow for non-zero correlations between instruments and disturbances.

Using

$$\Sigma_x = diag(\sigma_1, ..., \sigma_K) \tag{2.8}$$

(2.7) implies

$$E(x_i u_i) = \sigma_{xu} = E[x_i(y_i - x_i'\beta)] = E(x_i y_i) - E(x_i x_i')\beta = \sigma_u \Sigma_x r.$$

Invoking again the "analogy principle" this suggests for the method of moments estimator the solution $\hat{\beta}$, where

$$n^{-1} X'y - n^{-1} X'X\hat{\beta} = \sigma_u S_x r.$$

Here S_x is the sample equivalent of Σ_x. The j-th diagonal element of S_x could either be taken as the square root of $n^{-1}\Sigma_{i=1}^n x_{ij}^2$ (since the regressors have zero expectation) or $(n-1)^{-1}\Sigma_{i=1}^n (x_{ij} - \bar{x}_j)^2$ with $\bar{x}_j = n^{-1}\Sigma_{i=1}^n x_{ij}$ (which may be beneficial in small samples where $\bar{x}_j$ may deviate seriously from zero). This yields solution

$$\begin{aligned} \hat{\beta}(r, \sigma_u) &= (X'X)^{-1} X'y - \sigma_u (n^{-1} X'X)^{-1} S_x r \\ &= \hat{\beta}_{OLS} - \sigma_u (n^{-1} X'X)^{-1} S_x r. \end{aligned} \tag{2.9}$$

This estimator involves a correction to the OLS estimator, aiming to correct its inconsistency when $\rho_{xu} \neq 0$.

Estimator $\hat{\beta}(r, \sigma_u)$ is unfeasible as long as σ_u has not been replaced by a sample equivalent. Of course, the standard OLS estimator of σ_u^2, which is given by $\hat{\sigma}^2_{u,OLS} = \hat{u}'_{OLS}\hat{u}_{OLS}/(n-K)$, where $\hat{u}_{OLS} = y - X\hat{\beta}_{OLS}$, is inconsistent, like $\hat{\beta}_{OLS}$, when $\rho_{xu} \neq 0$, since

$$\begin{aligned}\operatorname{plim} \hat{\sigma}^2_{u,OLS} &= \operatorname{plim} u'[I - X(X'X)^{-1}X']u/n \\ &= \sigma_u^2 - \operatorname{plim} n^{-1}u'X(\operatorname{plim} n^{-1}X'X)^{-1} \operatorname{plim} n^{-1}X'u \\ &= \sigma_u^2(1 - \rho'_{xu}\Sigma_x\Sigma_{xx}^{-1}\Sigma_x\rho_{xu}).\end{aligned}$$

Thus, a feasible though r-based estimator, which attempts to correct $\hat{\sigma}^2_{u,OLS}$ for its inconsistency, is

$$\hat{\sigma}_u^2(r) = \hat{\sigma}^2_{u,OLS}/(1 - r'S_xS_{xx}^{-1}S_xr), \tag{2.10}$$

where element j,k of S_{xx} equals either $n^{-1}\Sigma_{i=1}^n x_{ij}x_{ik}$ or $(n-1)^{-1}\Sigma_{i=1}^n(x_{ij}-\bar{x}_j)(x_{ik}-\bar{x}_k)$. Now a feasible estimator which attempts to correct $\hat{\beta}_{OLS}$ for its inconsistency is

$$\hat{\beta}(r) = \hat{\beta}_{OLS} - \hat{\sigma}_u(r)S_{xx}^{-1}S_xr. \tag{2.11}$$

It is obvious that the correction terms of (2.10) and (2.11) only really succeed in discarding the OLS estimators from their inconsistency if $r = \rho_{xu}$.

3. KLS inference for multiple regressions

In order to examine the expedience of estimator (2.11) for producing inference on β we shall first examine its limiting distribution under the (unrealistic) assumption that the K values r equal the true correlations ρ_{xu}. Under the assumptions made above, it is found that consistent (though unfeasible) least-squares estimator $\hat{\beta}(\rho_{xu})$ has a limiting normal distribution with in general a rather involved variance matrix. Its asymptotic variance appears to be affected by the skewness and kurtosis of the distributions of u_i and x_i. Substantial simplifications occur when the third and fourth moments correspond to those of the normal distribution and especially in models with just one endogenous explanatory variable a remarkably neat result emerges.

For the unfeasible estimator $\hat{\beta}(\rho_{xu})$, which generalizes for multiple structural regression models the KLS estimator of Kiviet (2013, Section 4), we find the following result (proof in Appendix B) for models with an arbitrary number of endogenous regressors, where all regressors and disturbances are identically distributed and have their first four moments similar to normally distributed variables.

Theorem 1: *In zero mean IID cross-section model (2.1), where* $E(x_iu_i) = \rho_{xu}$, *while* $E(u_i^3) = 0$, $E(u_i^4) = 3\sigma_u^4$, $E(x_{ij}^3) = 0$ *and* $E(x_{ij}^4) = 3\sigma_j^4$ *for* $j = 1, ..., K$ *and* $i = 1, ..., n$, *we find for* $\hat{\beta}(\rho_{xu}) = \hat{\beta}_{OLS} - \hat{\sigma}_u(\rho_{xu})S_{xx}^{-1}S_x\rho_{xu}$ *the limiting distribution*

$$n^{1/2}[\hat{\beta}(\rho_{xu}) - \beta] \xrightarrow{d} \mathcal{N}[0, \sigma_u^2 V(\rho_{xu})],$$

with $V(\rho_{xu}) = \Sigma_{xx}^{-1}\Theta\Sigma_{xx}^{-1}$, *where*

$$\begin{aligned}\Theta = \Sigma_{xx} - \Sigma_{xx}\mathcal{R}^2 - \mathcal{R}^2\Sigma_{xx} - \theta^{-1}(\Sigma_{xx}\mathcal{R}^2\Sigma_{xx}^{-1}\Phi + \Phi\Sigma_{xx}^{-1}\mathcal{R}^2\Sigma_{xx}) \\ - 0.5\theta^{-1}(\Phi\mathcal{R}^2 + \mathcal{R}^2\Phi) + [\theta^{-1} + \theta^{-2}(0.5 - \rho'_{xu}\Sigma_x\Sigma_{xx}^{-1}\Sigma_x\mathcal{R}^2\rho_{xu})]\Phi \\ + 0.5(I + \theta^{-1}\Phi\Sigma_{xx}^{-1})\mathcal{R}\Sigma_x^{-1}(\Sigma_{xx} \circ \Sigma_{xx})\Sigma_x^{-1}\mathcal{R}(I + \theta^{-1}\Sigma_{xx}^{-1}\Phi).\end{aligned}$$

Here $\mathcal{R} = \text{diag}(\rho_{xu})$ *is the diagonal matrix with the elements of* ρ_{xu} *on its main diagonal,* $\theta = 1 - \rho'_{xu}\Sigma_x\Sigma_{xx}^{-1}\Sigma_x\rho_{xu}$ *and* $\Phi = \Sigma_x\rho_{xu}\rho'_{xu}\Sigma_x$.

The following corollaries readily follow.

Corollary 1.1: *If in the situation of Theorem 1 one has* $K = 1$, *thus* $\Sigma_{xx} = \sigma_x^2$ *and* $\mathcal{R} = \rho_{xu}$ *are scalar, then* $\Theta = \sigma_x^2$ *so that* $V(\rho_{xu}) = \sigma_x^{-2}$ *is invariant with respect to* ρ_{xu}, *and* $n^{1/2}[\hat{\beta}(\rho_{xu}) - \beta] \xrightarrow{d} \mathcal{N}(0, \sigma_u^2/\sigma_x^2)$.

A proof of Corollary 1.1 can already be found in Kiviet (2013).

Another interesting special case of Theorem 1 considers the situation where just one regressor (say the first one) is endogenous ($K_1 = 1$) and all further regressors are exogenous. This leads to the following.

Corollary 1.2: *If in the situation of Theorem 1* $\rho_{xu} = (\rho_1, 0, ..., 0)'$ *then, denoting* $\hat{\beta}(\rho_{xu})$ *now as* $\hat{\beta}(\rho_1)$, *we have* $n^{1/2}[\hat{\beta}(\rho_1) - \beta] \xrightarrow{d} \mathcal{N}[0, \sigma_u^2 V(\rho_1)]$ *with*

$$V(\rho_1) = \Sigma_{xx}^{-1} - \theta^{-1}\rho_1^2\{e_1e_1'\Sigma_{xx}^{-1} + \Sigma_{xx}^{-1}e_1e_1' - [1 + \theta^{-1}(1 - \rho_1^2)]\sigma_1^2\Sigma_{xx}^{-1}e_1e_1'\Sigma_{xx}^{-1}\},$$

where $\theta = 1 - \rho_1^2\sigma_1^2\sigma^{11}$ *with* $\sigma^{11} = (\Sigma_{xx}^{-1})_{1,1}$. *Moreover, for the first element of vector* $\hat{\beta}(\rho_1)$, *denoted* $\hat{\beta}_1(\rho_1)$, *this yields*

$$n^{1/2}[\hat{\beta}_1(\rho_1) - \beta_1] \xrightarrow{d} \mathcal{N}(0, \sigma_u^2\sigma^{11}),$$

which is invariant with respect to ρ_1.

Note that surprisingly the limiting distribution of the coefficient of the one and only endogenous regressor of the inconsistency corrected least-squares estimator (KLS), irrespective of the occurrence in the model of any further exogenous regressors, is equivalent to that of OLS in case all regressors are exogenous. This is no longer the case when $K_1 > 1$.

To produce feasible inference the result of Theorem 1 can be exploited as follows. Suppose we are interested in testing jointly h ($\leq K$) linear restrictions on the coefficients β of model (2.1), given by $\mathcal{H}_0 : Q\beta = q$, where Q is a known $h \times K$ matrix of rank h and q a $h \times 1$ known vector. Now consider the feasible test statistic

$$W(Q, q, r) = [Q\hat{\beta}(r) - q]'[n^{-1}Q\hat{V}(r)Q']^{-1}[Q\hat{\beta}(r) - q]/\hat{\sigma}_u^2(r), \tag{3.1}$$

where $\hat{V}(r) = S_{xx}^{-1}\hat{\Theta}S_{xx}^{-1}$, with

$$\begin{aligned}\hat{\Theta} = S_{xx} - S_{xx}R^2 - R^2S_{xx} - \hat{\theta}^{-1}(S_{xx}R^2S_{xx}^{-1}\hat{\Phi} + \hat{\Phi}S_{xx}^{-1}R^2S_{xx}) \\ - 0.5\hat{\theta}^{-1}(\hat{\Phi}R^2 + R^2\hat{\Phi}) + [\hat{\theta}^{-1} + \hat{\theta}^{-2}(0.5 - r'S_xS_{xx}^{-1}S_xR^2r)]\hat{\Phi} \\ + 0.5(I + \hat{\theta}^{-1}\hat{\Phi}S_{xx}^{-1})RS_x^{-1}(S_{xx} \circ S_{xx})S_x^{-1}R(I + \hat{\theta}^{-1}\Sigma_{xx}^{-1}\hat{\Phi}),\end{aligned}$$

and $\hat{\theta} = 1 - r'S_x S_{xx}^{-1} S_x r$, $\hat{\Phi} = S_x r r' S_x$ and $R = \text{diag}(r)$. Under $\mathcal{H}_0$ and the conditions of Theorem 1 we have

$$W(Q, q, \rho_{xu}) \xrightarrow{d} \chi^2(h). \tag{3.2}$$

Now for $r = (\gamma', 0')'$ and $\chi^2_{1-\alpha}(h)$ the $(1-\alpha) \times 100\%$ quantile of the chi-squared distribution with h degrees of freedom, the set

$$\mathcal{C}_0(Q, q, \alpha) = \left\{\gamma \in \mathbb{R}^{K_1} : T(Q, q, r) < \chi^2_{1-\alpha}(h)\right\} \tag{3.3}$$

represents all possible values of $\rho_{x^{(1)}u}$ for which $\mathcal{H}_0$ does not have to be rejected at asymptotic significance level α. Likewise, the compliment of $\mathcal{C}_0(Q, q, \alpha)$ in $\mathbb{R}^{K_1}$, given by $\mathcal{C}_1(Q, q, \alpha) \equiv \mathbb{R}^{K_1} \backslash \mathcal{C}_0(Q, q, \alpha)$, represents all possible values of $\rho_{x^{(1)}u}$ for which $\mathcal{H}_0$ should be rejected at asymptotic significance level α.

The sets $\mathcal{C}_0(Q, q, \alpha)$ and $\mathcal{C}_1(Q, q, \alpha)$ enable to supplement standard OLS inference on $\mathcal{H}_0$ with an indication of its robustness or sensitivity regarding simultaneity. Suppose that the $K_1 \times 1$ zero vector is in set $\mathcal{C}_0(Q, q, \alpha)$, then this set represents also all non-zero values of $\rho_{x^{(1)}u}$ which corroborate under simultaneity the non-rejection of $\mathcal{H}_0$ established under full exogeneity. The OLS decision not to reject $\mathcal{H}_0$ is robust regarding simultaneity as long as it obeys the restrictions set by $\mathcal{C}_0(Q, q, \alpha)$, whereas for values of $\rho_{x^{(1)}u}$ in $\mathcal{C}_1(Q, q, \alpha)$ $\mathcal{H}_0$ should be rejected. When the zero vector is in set $\mathcal{C}_1(Q, q, \alpha)$, thus standard OLS inference rejects $\mathcal{H}_0$, this decision can be extended under simultaneity represented by all vectors $\rho_{x^{(1)}u}$ in $\mathcal{C}_1(Q, q, \alpha)$, but should be reversed for values of $\rho_{x^{(1)}u}$ in $\mathcal{C}_0(Q, q, \alpha)$.

It is obvious that the inference based on KLS as just described will be fully robust with respect to simultaneity only if $\mathcal{C}_0(Q, q, \alpha)$ is $\mathbb{R}^{K_1}$ or $\varnothing$. This seems highly unlikely, as becomes clear when we examine the case $K_1 = 1$ as in Kiviet (2013). It just extends the asymptotic validity of standard OLS for the case $\rho_{x^{(1)}u} = 0$ to asymptotic validity for either $\rho_{x^{(1)}u} \in \mathcal{C}_0(Q, q, \alpha)$ or $\rho_{x^{(1)}u} \in \mathcal{C}_1(Q, q, \alpha)$. This could be labelled constraint robustness.

Of course, the actual numerical assessment and representation of the above sets may be quite complicated in practice, especially when K_1 is (much) larger than 1, and when the number of tested restrictions h is too. In the special case $K_1 = 1 \leq K$ and $h = 1$, while the tested restriction concerns just the coefficient of the endogenous regressor, it is actually quite simple, as already exposed in Kiviet (2013, 2016). In more general cases obvious problems regarding the numerical feasibility of this approach will occur in samples, and choices of r, where the correction of $\hat{\sigma}_u^2$ does no longer make sense, which is the case when $r'S_x S_{xx}^{-1} S_x r \geq 1$. We will monitor this in Sections 5 and 6.

4. Testing exclusion restrictions

One of the paradigms of classic econometric theory is that the exclusion restrictions in a just identified model cannot be tested, and that in overidentified models one cannot test all exclusion restrictions but just a limited number of them equal to the degree of overidentification. Hence, in both cases some exclusion restrictions seem non-testable.

By the methodology exposed above, however, it is possible to a certain extent to test any exclusion restrictions. It enables to assess, at a chosen nominal significance level, the set of all possible ρ_{xu} values for which any arbitrary subset of exclusion restrictions should be rejected. If this set seems to cover the area in which the true value of ρ_{xu} may

reside, then one should reject validity of the variables associated with these exclusion restrictions as instruments. On the other hand, when it seems likely that the true value of ρ_{xu} will not be in the assessed set, or when this set is empty, rejection of validity of the instruments under test is not indicated. Hence, at the stage of deciding whether or not the true value of ρ_{xu} seems covered by a particular non-empty set, expert knowledge is required to decide on the validity or not of an instrument, as in the case regarding adopting $\rho_{zu} = 0$ or not. However, the assessed set regarding ρ_{zu} may turn out to be so wide (or so narrow) that the decision becomes relatively easy. By calculating P-values of $T(Q, q, r)$ for deliberately chosen Q and r and all relevant values of r, we will show in the illustrations below how evidence on the (in)validity of instruments can be produced which in many cases may be more convincing than evidence just based on pure rethoric arguments.

Note that the procedure just sketched is not an alternative to the test for overidentifying restrictions, often addressed as Sargan-Hansen test. This test presupposes validity of a number of external instruments equal to the number of endogenous regressors in the model, which just-identify the model, and then tests the validity of additional overidentifying instruments. The procedure discussed here can be implemented such that it produces inference on the validity of any set of candidate instruments, so also on this initial set on which standard (and incremental or difference) overidentifying restrictions tests build without any prior statistical verification.

Here we will first work out in detail this test for just-identifying exclusion restrictions for the model introduced in section 2.1 focussing on the special case $K_1 = 1$, whereas $L = K$. Hence, the structural multiple regression model is just identified, x_{i1} is the one and only endogenous regressor, and the question is whether external scalar variable z_{i2} is exogenous and thus can be used as an instrument next to the $K-1$ regressor variables $x_i^{(2)}$, which are maintained to be exogenous. Hence, we will test the validity of z_{i2} as an instrument, or the assumption $E(z_{i2}u_i) = 0$. This is untestable by the Sargan-Hansen approach, because it requires $L > K$.

Attempting to test $E(z_{i2}u_i) = 0$ could be done by including z_{i2} in the regression and test by an appropriate method whether its coefficient is significant. Its insignificance would endorse (although certainly not guarantee) its valid exclusion from the regression and use as an external instrument. To test by the established methods in model[4]

$$y_i = x_{i1}\beta_1 + x_i^{(2)\prime}\beta_2 + z_{i2}\beta_z + u_i, \tag{4.1}$$

the exclusion hypothesis $\mathcal{H}_e : \beta_z = 0$, while respecting at the same time the simultaneity of regressor x_{i1}, would require yet another valid external instrument, which would bring their number to $L + 1 = K + 1$, whereas we assumed that, apart from the $K - 1$ exogenous regressors $x_i^{(2)}$, the only further candidate instrument is z_{i2}. So, testing the exclusion restriction $\beta_z = 0$ seems impossible indeed.

Though, in the present situation, the first result of Corollary 1.2 applies, after translating it from model (2.1) into the context of augmented model (4.1). The latter we will denote as $y = X^*\beta^* + u$, where $X^* = (X, z_2)$ and $\beta^* = (\beta', \beta_z)'$. With $1 \times (K+1)$ matrix $Q = (0, ..., 0, 1) = e'_{K+1}$ and $q = 0$, and r_1 still indicating the assumed value for ρ_{x_1u}, we

[4]In correspondence with our set-up in Section 2, we may assume that the arbitrary intercept in the relationship has been partialled out, from y_i, x_{i1}, $x_i^{(2)}$ and z_{i2}, by taking these sample values in deviation from their sample mean.

may use for this single hypothesis the test statistic

$$t_{K+1}(r_1) = \hat{\beta}_z(r_1)/[n^{-1}\hat{\sigma}_u^{*2}(r_1)e'_{K+1}\hat{V}^*(r_1)e_{K+1}]^{1/2}, \tag{4.2}$$

where, when using the notation $S^{-1}_{x^*x^*} = (n^{-1}X^{*\prime}X^*)^{-1}$, and e_1 now representing the unit vector with $K+1$ elements, we have

$$\hat{\beta}_z(r_1) = e'_{K+1}(\hat{\beta}^* - \hat{\sigma}_u^*(r_1)S^{-1}_{x^*x^*}e_1 s_1 r_1) = \hat{\beta}_z - \hat{\sigma}_u^*(r_1)s_1 r_1(e'_{K+1}S^{-1}_{x^*x^*}e_1) \tag{4.3}$$

and

$$e'_{K+1}\hat{V}^*(r_1)e_{K+1} = e'_{K+1}S^{-1}_{x^*x^*}e_{K+1} + r_1^2[1+(1-r_1^2)/\hat{\theta}^*(r_1)]s_1^2(e'_1S^{-1}_{x^*x^*}e_{K+1})^2/\hat{\theta}^*(r_1), \tag{4.4}$$

with

$$\hat{\beta}_z = [z'_2(I-P_X)z_2]^{-1}z'_2(I-P_X)y, \tag{4.5}$$

$$\hat{\theta}^*(r_1) = 1 - r_1^2 s_1^2(e'_1S^{-1}_{x^*x^*}e_1), \tag{4.6}$$

$$\hat{\sigma}_u^{*2}(r_1) = \tilde{n}^{-1}y'(I-P_{X^*})y/\hat{\theta}^*(r_1), \tag{4.7}$$

where $\tilde{n}$ could simply be chosen n, or if one wants to employ a small sample adjustment it could be taken $n-K-1$ or, when all variables have been taken in deviation from their sample average, $n-K-2$.

It is easy to show that θ of Theorem 1 is always positive. To achieve this for $\hat{\theta}^*(r_1)$ too, we should not vary r_1 over the whole $(-1,+1)$ interval, but just examine $r_1^2 < (e'_1S^{-1}_{x^*x^*}e_1)/s_1^2$. However, also in samples where $\hat{\theta}^*(r_1)$ happens to be positive but much smaller than its true value θ^* estimators $\hat{\sigma}_u^{*2}(r_1)$, $\hat{\beta}_z(r_1)$ and its estimated standard deviation may be seriously affected. This may lead to unpleasant consequences for the distribution of the test statistic. Such consequences seem more likely when n is small and when θ^* is small. We will monitor this in the simulations in the next Section.

Extending the assumptions of Theorem 1 to model (4.1) and evaluating (4.2) in ρ_1, we have $t_{K+1}(\rho_1) \xrightarrow{d} \mathcal{N}(0,1)$ under $\mathcal{H}_e$. Hence, if ρ_1 were known an asymptotically exact test would be available. If ρ_1 is unknown, and when testing two-sided, we should seek the set of all r_1 values for which $[t_{K+1}(r_1)]^2 > \chi^2_{1-\alpha}(1)$ or

$$[\hat{\beta}_z(r_1)]^2 - [n^{-1}\hat{\sigma}_u^{*2}(r_1)e'_{K+1}\hat{V}^*(r_1)e_{K+1}] \times \chi^2_{1-\alpha}(1) > 0. \tag{4.8}$$

The left hand side of this inequality is non-linear in the scalar r_1. Assuming that finding the roots of $[\hat{\beta}_z(r_1)]^2 - [n^{-1}\hat{\sigma}_u^{*2}(r_1)e'_{K+1}\hat{V}^*(r_1)e_{K+1}] \times \chi^2_{1-\alpha}(1) = 0$ over domain $|r_1| < 1$ is feasible, finding the set of r_1 values for which inequality (4.8) holds will be feasible too. Under the assumption that the true value of ρ_1 is contained in this set, the hypothesis $\rho_{z^{(2)}u} = 0$ should be rejected. This procedure has an asymptotic significance level not larger than α. Small sample performance may improve upon replacing $\chi^2_{1-\alpha}(1)$ by $F_{1-\alpha}(1,\tilde{n})$.

Instead of finding the roots of (4.8) at a particular α an easier and more informative approach is to construct a graph over all relevant values of r_1, satisfying $r_1^2 < (e'_1S^{-1}_{x^*x^*}e_1)/s_1^2$, of the P-values of $t^2_{K+1}(r_1)$ with respect to the $F(1,\tilde{n})$ distribution. For any α this immediately shows the range of values for ρ_{x_1u} where the test statistic rejects (or not) the exclusion restriction.

Of course, for any $L_2^* \times 1$ subset z_i^* from $L_2 \times 1$ vector $z_i^{(2)}$ its valid exclusion from model (2.1) can be tested in a similar way, also in models where $K_1 \geq 1$. This involves a special implementation of test (3.1). Now let $X^* = (X, Z^*)$ and $\beta^* = (\beta', \beta'_{z^*})'$ with $\hat{\beta}_{z^*} = (Z^{*\prime} M_X Z^*)^{-1} Z^{*\prime} M_X y$ whereas $Q^* = (O, I_{L_2^*})$. Consider test statistic

$$W^*(Q^*, r) = \hat{\beta}'_{z^*}[n^{-1} Q^* \hat{V}^*(r) Q^{*\prime}]^{-1} \hat{\beta}_{z^*} / [L_2^* \times \hat{\sigma}_u^2(r)], \tag{4.9}$$

where $\hat{V}^*(r) = S^{-1}_{x^*x^*} \hat{\Theta}^* S^{-1}_{x^*x^*}$, with $\hat{\Theta}^*$ the appropriate adaptation of $\hat{\Theta}$ below (3.1) to the present extended model. When $E(z_i^* u_i) = 0$ then $L_2^* \times W^*(Q^*, \rho_{xu}) \xrightarrow{d} \chi^2(L_2^*)$. Calculating the P-value of $W^*(Q^*, r)$ with respect to the $F(L_2^*, \tilde{n})$ distribution over all relevant values r indicates for which values ρ_{xu} validity of the instruments z_i^* seems (un)likely.

5. Simulation results

In this section we want to produce simulation evidence on the finite sample behavior of the inference techniques suggested in this study. As always such a study is only feasible when one strongly restrains the number of parameters of the simulation design. For practical reasons one also has to constrain the grid of parameter values from the design parameter space for which the techniques are actually examined. Therefore simulated models do often not fully mimic all aspects of empirically relevant models, but just their major basic characteristics. However, sometimes one can prove that the phenomenon of interest is in fact invariant with respect to particular parameters, which implies that relatively few calculations for a discrete choice of parameter values can represent the relevant properties for a whole subspace of the design parameter space. The model introduced in Section 2 is primarily characterized by the values n, K_1, K_2 and L_2, where $L_2 \geq K_1 \geq 1$, $L_1 = K_2 \geq 0$ and $n \gg L_1 + L_2 \geq K_1 + K_2$, and also by β, Σ_{xx}, Σ_{zz}, Σ_{zx}, σ_u^2, ρ_{xu} and ρ_{zu} In Kiviet (2013) very favorable results have been produced on the finite sample accuracy of KLS inference on β in the very simple case $L_2 = K_1 = 1$ with $L_1 = K_2 = 0$ when ρ_{xu} were known. But, also for the more realistic situation where ρ_{xu} is unknown and an (in)correct interval $[\rho_{xu}^L, \rho_{xu}^U]$ is adopted which is supposed to contain the true value, it is shown that KLS inference is often much more useful than standard or Anderson-Rubin instrument-based inference. Because one may suppose that the simulated data in these experiments have been obtained after partialling out any exogenous regressors, the results are invariant regarding the chosen value for K_2, so they represent the situation for any $K \geq K_1 = 1$. For this situation, also the actually chosen values for β, Σ_{xx}, Σ_{zz} and σ_u^2 have been shown not to affect this KLS based test.

Below, in the first subsection, we will consider the same simulation design as used in Kiviet (2013), but examine now the finite sample behavior of the exclusion restriction test, under situations where ρ_{xu} is either known or unknown. Next we will present simulation results on KLS inference regarding β and exclusion restrictions tests for models where $K \geq K_1 = L_2 = 2$. All presented results are based on 250,000 replications.

5.1. The simplest possible implementation of the exclusion restriction test

For the very simple model with $K = K_1 = L = L_2 = 1$ we will examine here some of the small sample qualities of test (4.2) on a single just-identifying exclusion restriction. Due

to invariance, the results will also represent cases where $L_1 = K_2 > 0$ and $K = L > 1$. The Monte Carlo design is constructed as follows. Let ε_i, ξ_i and ζ_i be three mutually independent series $(i = 1, ..., n)$ of standard normal drawings. From these we construct the three series

$$u_i = \sigma_u \varepsilon_i \sim \mathcal{N}(0, \sigma_u^2), \tag{5.1}$$

$$x_i = \sigma_x[(1 - \rho_{xu}^2)^{1/2} \xi_i + \rho_{xu} \varepsilon_i] \sim \mathcal{N}(0, \sigma_x^2), \tag{5.2}$$

$$z_i = \sigma_z(\rho_{z\zeta} \zeta_i + \rho_{z\xi} \xi_i + \rho_{zu} \varepsilon_i) \sim \mathcal{N}(0, \sigma_z^2), \tag{5.3}$$

where all ρ coefficients do not exceed 1 in absolute value; moreover,

$$\rho_{z\zeta}^2 + \rho_{z\xi}^2 + \rho_{zu}^2 = 1. \tag{5.4}$$

Obviously, $\sigma_{xu} = \rho_{xu}\sigma_x\sigma_u$, $\sigma_{zu} = \rho_{zu}\sigma_z\sigma_u$ and $\sigma_{zx} = \sigma_z\sigma_x[\rho_{z\xi}(1 - \rho_{xu}^2)^{1/2} + \rho_{zu}\rho_{xu}]$, hence $\rho_{zx} = \rho_{z\xi}(1 - \rho_{xu}^2)^{1/2} + \rho_{zu}\rho_{xu}$, which yields

$$\rho_{z\xi} = (\rho_{zx} - \rho_{zu}\rho_{xu})(1 - \rho_{xu}^2)^{-1/2}, \tag{5.5}$$

for $\rho_{xu}^2 < 1$. From (5.4) we also have

$$\rho_{z\zeta} = (1 - \rho_{z\xi}^2 - \rho_{zu}^2)^{1/2}. \tag{5.6}$$

Hence, when values for $\sigma_u > 0$, $\sigma_x > 0$, $\sigma_z > 0$, $|\rho_{xu}| < 1$, $|\rho_{zx}| \leq 1$ and $|\rho_{zu}| \leq 1$ are chosen, we can generate the series for u_i and x_i and find matching values for $\rho_{z\xi}$ from (5.5) and for $\rho_{z\zeta}$ from (5.6) so that series z_i can be generated as well. However, the three chosen correlations should obey

$$(\rho_{zx} - \rho_{zu}\rho_{xu})^2 \leq (1 - \rho_{xu}^2)(1 - \rho_{zu}^2), \tag{5.7}$$

in order to ensure that $0 \leq \rho_{z\xi}^2 \leq 1$ and $0 \leq \rho_{z\zeta}^2 \leq 1$.

For each realization of the series u_i, x_i and z_i in the simulation replications, we may first subtract their respective sample average from each observation. In that way an arbitrary intercept of an underlying model with one further regressor and one external potential instrument (each distributed with a possibly non-zero arbitrary mean) has been partialled out.

The dependent variable is generated by the model

$$y_i = x_i\beta + z_i\beta_z + u_i, \tag{5.8}$$

where coefficient β_z has true value zero. Its standard least-squares estimator (4.5) simplifies to

$$\hat{\beta}_z = \frac{z'u - z'x(x'x)^{-1}x'u}{z'z - z'x(x'x)^{-1}z'x} = \frac{r_{zu} - r_{zx}r_{xu}}{1 - r_{zx}^2}\frac{s_u}{s_z}, \tag{5.9}$$

where we define the sample statistics as $r_{z'u} = z'u/(z'zu'u)^{1/2}$, $s_u = | \sqrt{s_u^2} |$ with $s_u^2 = u'u/n_1$ (and similar for r_{zx}, r_{xu}, s_x and s_z), where n_1 is either $n - 1$ or n, depending on whether deviations from sample average have been taken or not.

For this special model we further have from (4.6), taking r_1 as appraisal of $\rho_1 = \rho_{xu}$,

$$\hat{\theta}^*(r_1) = 1 - \frac{r_1^2 s_x^2 s_z^2}{s_x^2 s_z^2 (1 - r_{zx}^2)} = \frac{1 - r_1^2 - r_{zx}^2}{1 - r_{zx}^2}, \tag{5.10}$$

and, because $y'(I-P_{X^*})y \geq 0$ specializes here to $u'u[1-(r_{xu}^2-2r_{xu}r_{zx}r_{zu}+r_{zu}^2)/(1-r_{zx}^2)]$, we find for (4.7) the expression

$$\hat{\sigma}_u^{*2}(r_1) = \frac{u'u}{\tilde{n}} \frac{1 - r_{zx}^2 - r_{zu}^2 + 2r_{zx}r_{zu}r_1 - r_1^2}{1 - r_1^2 - r_{zx}^2}, \tag{5.11}$$

which will be positive (as a variance estimate should) provided $r_1^2+r_{zx}^2 < 1$ or $\hat{\theta}^*(r_1) > 0$. Clearly, practical problems may emerge in cases where r_1 is chosen large in absolute value and r_{zx}^2 happens to be larger than ρ_{zx}^2. In the simulations we will monitor the occurrence of $\hat{\theta}^*(r_1) \leq 0$ (which may be frequent, especially when n is small and the variance of r_{zx} large) but will skip such replications, because $\hat{\beta}_z(r_1)$ is only defined when $\hat{\theta}^*(r_1) > 0$. In this simple model it specializes to

$$\begin{aligned}\hat{\beta}_z(r_1) &= \hat{\beta}_z + \hat{\sigma}_u^*(r_1) \frac{r_1 r_{zx}}{1 - r_{zx}^2} \frac{1}{s_z} \\ &= \frac{r_{zu} - r_{zx}r_1 \left[1 - \left(\frac{n_1(1-r_{zx}^2-r_{zu}^2+2r_{zx}r_{zu}r_1-r_1^2)}{\tilde{n}(1-r_1^2-r_{zx}^2)}\right)^{1/2}\right]}{1 - r_{zx}^2} \frac{s_u}{s_z}.\end{aligned} \tag{5.12}$$

For its estimated asymptotic variance, assuming $r_1 = \rho_1$, we find

$$\frac{n_1}{\tilde{n}n} \frac{1 - r_{zx}^2 - r_{zu}^2 + 2r_{zx}r_{zu}r_1 - r_1^2}{(1 - r_{zx}^2)^2 \hat{\theta}^*(r_1)} \left[1 + r_1^2 r_{zx}^2 [1 + (1 - r_1^2)/\hat{\theta}^*(r_1)]/\hat{\theta}^*(r_1)\right] \frac{s_u^2}{s_z^2}. \tag{5.13}$$

From the expressions (5.12), (5.13) and (5.10) we observe that in this special model both $\hat{\beta}_z(r_1)$ and its asymptotic standard error are invariant with respect to β and to s_x, whereas both are a multiple of s_u/s_z. Hence, in this simple model the exclusion restriction test statistic (4.2) will be invariant to β, σ_u^2, σ_x^2 and σ_z^2. Therefore, without loss of generality, we may set in the simulation: $\beta = 0$ and $\sigma_u = \sigma_x = \sigma_z = 1$. Another invariance result is the following. If $r_1 = \rho_{xu}$ and from two of the three correlations ρ_{xu}, ρ_{zx} and ρ_{zu} we change their sign, then the square of the exclusion test statistic does not change. Hence, considering in the simulations only cases where these three correlations are nonnegative (as we will) is not as restrictive as it seems at first sight.

We also find

$$\operatorname{plim} \hat{\beta}_z(r_1) = \frac{\rho_{zu} - \rho_{zx}r_1 \left[1 - \left(1 + \rho_{zu}\frac{2\rho_{zx}r_1 - \rho_{zu}}{1-r_1^2-\rho_{zx}^2}\right)^{1/2}\right]}{1 - \rho_{zx}^2} \frac{\sigma_u}{\sigma_z},$$

which is zero when $\rho_{zu} = 0$. Because it seems to be mostly non-zero for $\rho_{zu} \neq 0$, we are hopeful that a test based on KLS estimator $\hat{\beta}_z(r_1)$ may have power for testing the invalidity of instrument z_i for the regression of y_i on x_i. From (5.9) we find that the pseudo-true-value of $\hat{\beta}_z$ is

$$\operatorname{plim} \hat{\beta}_z = \frac{\rho_{zu} - \rho_1\rho_{zx}}{1 - \rho_{zx}^2} \frac{\sigma_u}{\sigma_z},$$

which is non-zero in general, unless $\rho_{zu} = \rho_1\rho_{zx}$. Hence, even when $\rho_{zu} = 0$ it may be non-zero, unless also $\rho_1 = 0$ or $\rho_{zx} = 0$. So obviously, the exclusion restriction should not be tested on the basis of the standard OLS estimator $\hat{\beta}_z$.

In Tables 1 ($n = 500$) and 2 ($n = 50$) we report the rejection frequency of the two-sided unfeasible test based on the square of statistic (4.2), where we substituted the true value of ρ_1 for r_1. Since we took the generated data series in deviation from their sample mean we used $n_1 = n - 1$ and, employing the 5% nominal critical value of the F-distribution, we took it at 1 and $n - 3$ degrees of freedom.

Table 1: Rejection frequency (in %) of the unfeasible exclusion restriction test ($n = 500; \alpha = 0.05$)

	$\rho_{zu} = 0$			$\rho_{zu} = 0.05$			$\rho_{zu} = 0.1$			$\rho_{zu} = 0.2$		
ρ_{xu}	ρ_{zx}			ρ_{zx}			ρ_{zx}			ρ_{zx}		
	0.0	0.4	0.8	0.0	0.4	0.8	0.0	0.4	0.8	0.0	0.4	0.8
0.0	5.09	5.09	5.09	20.1	23.0	46.2	61.0	68.8	96.3	99.4	99.9	100
0.2	5.09	5.07	5.01	20.1	22.7	43.9	61.0	67.9	95.1	99.4	99.8	100
0.4	5.05	4.99	4.68	20.1	22.1	37.2	61.1	66.8	92.7	99.5	99.8	100
0.6	5.02	4.86	100#	20.0	20.4	0.15#	61.2	64.5	0.00#	99.5	99.7	6.09#
0.8	4.84	4.14	-	19.9	10.7	-	61.7	53.4	-	99.6	99.5	-

In the block of results for $\rho_{zu} = 0$ we observe in Table 1 that the asymptotic test using the true value of ρ_1 demonstrates very good size control[5] when $n = 500$. According to inequality (5.7) the model is not defined for cases where ρ_{zu} is moderate and both ρ_{xu} and ρ_{zx} are large in absolute value (indicated by "-" in the tables). As already predicted, for cases where $\rho_{xu}^2 + \rho_{zx}^2$ is close to unity we observe deterioration of the performance of the test. Settings for which some experiments did produce negative $\hat{\theta}^*(\rho_1)$ realizations are indicated by a hashtag. In fact, undefinedness of KLS occurred for those cases in about 50% of the replications. At this large sample size the power of the test is already remarkable for $\rho_{zu} = 0.05$, impressive for $\rho_{zu} = 0.1$ and outright splendid for $\rho_{zu} \geq 0.2$. Apart from close to the non existence region the rejection probability is found to be almost invariant with respect to the degree of simultaneity ρ_{xu}. For $\rho_{zu} \neq 0$ the rejection probability increases with the absolute value of ρ_{zx}, but is also very good for $\rho_{zx} = 0$, so the KLS exclusion restriction test does not suffer in any way from weak instrument problems.

Table 2: Rejection frequency (in %) of the unfeasible exclusion restriction test ($n = 50; \alpha = 0.05$)

	$\rho_{zu} = 0$			$\rho_{zu} = 0.1$			$\rho_{zu} = 0.2$			$\rho_{zu} = 0.3$		
ρ_{xu}	ρ_{zx}			ρ_{zx}			ρ_{zx}			ρ_{zx}		
	0.0	0.4	0.8	0.0	0.4	0.8	0.0	0.4	0.8	0.0	0.4	0.8
0.0	5.16	5.16	5.16	10.8	11.8	21.3	28.8	33.4	66.7	56.9	65.0	96.6
0.2	5.12	5.08	4.86	10.8	11.4	17.7	28.8	31.8	57.4	57.1	62.3	91.7
0.4	4.97	4.80	4.20*	10.6	10.2	1.98*	28.9	29.2	22.5*	57.5	58.9	67.1*
0.6	4.47	3.75	100#	10.2	5.54	1.54#	29.0	21.1	0.02#	58.8	50.6	0.03#
0.8	2.17	5.56*	-	7.87	0.64*	-	29.0	0.23*	-	63.7	4.41*	5.11#

Table 2 presents similar findings for sample size $n = 50$. Note that all results marked by an asteriks or hashtag have been obtained from fewer than 250,000 replications, because when $\hat{\theta}^*(\rho_1)$ turned out to be non-positive have been skipped (this occurred with frequency less than 5% for cases indicated by an asteriks and over 50% for cases indicated by a hashtag). Because in smaller samples r_{zx} may deviate much more from ρ_{zx} we note deterioration of the test qualities over a larger band of cases approaching the non existence area. Otherwise, however, the size properties of the test are still appropriate and power improves with the absolute value of ρ_{zu}, but self-evidently not as sharply as for larger samples.

In practice r_1 will usually deviate from ρ_1. Therefore, as in Kiviet (2013) for inference on β, we will now examine the merits of a feasible exclusion restriction test in this simple model when employed on the basis of an interval $[r_1^L, r_1^U]$ which is supposed to contain ρ_1.

[5] Given the number of replications used a probability of 5% will be estimated here with an error that could exceed ± 0.1 with a probability of about 2%.

We investigate the three cases $r_1^L = \rho_1 - 0.1$, $r_1^U = \rho_1 + 0.1$; $r_1^L = \rho_1 - 0.2$, $r_1^U = \rho_1 + 0.2$; and $r_1^L = 0$, $r_1^U = 0.3$. From Table 3 where $n = 100$ we see that when $\rho_{xu} \in [r_1^L, r_1^U]$ the test is undersized, and still has remarkable power away from the non existence region. The bottom two rows, where $\rho_{xu} \notin [r_1^L, r_1^U]$, show that the test can be either conservative or liberal. Far away from the non existence region the test may still help to produce useful inference on instrument (in)validity, but its results become uninterpretable otherwise. In this table the asteriks stands for a frequency to obtain an undefined test not exceeding 5% and a hashtag for a frequency exceeding 45%.

Table 3: Rejection frequency (in %) of the feasible exclusion restriction test ($n = 100; \alpha = 0.05$)

ρ_{xu}	r_{xu}^L	r_{xu}^U	$\rho_{zu} = 0$			$\rho_{zu} = 0.2$			$\rho_{zu} = 0.4$		
			ρ_{zx}			ρ_{zx}			ρ_{zx}		
			0.0	0.4	0.8	0.0	0.4	0.8	0.0	0.4	0.8
0.0	-0.1	0.1	4.14	1.72	0.10	48.1	41.5	53.4	98.3	98.5	100
0.2	0.1	0.3	4.10	1.63	0.09	47.8	41.5	49.6	98.2	98.4	100
0.4	0.3	0.5	3.99	1.39	0.05	46.6	39.8	21.1*	97.8	98.1	83.7*
0.6	0.5	0.7	3.66	0.80	100#	43.5	43.6	0.00*	96.9	97.7	0.02#
0.0	-0.2	0.2	3.19	0.50	0.00	43.7	24.7	9.30	97.7	95.0	99.4
0.2	0.0	0.4	3.17	0.44	0.00	42.9	26.2	9.88	97.3	95.7	98.9
0.4	0.2	0.6	3.00	0.31	0.00#	40.1	25.5	0.10*	95.7	95.9	21.5#
0.6	0.4	0.8	2.48	0.06*	100#	32.2	15.6*	0.00*	90.0	88.1*	0.00#
0.0	0.0	0.3	3.66	2.60	3.04	44.8	59.1	92.8	97.6	99.6	100
0.2	0.0	0.3	3.64	1.06	0.05	46.4	26.2	9.88	98.1	95.7	99.0
0.4	0.0	0.3	3.63	7.67	42.2	52.2	6.78	5.38	99.2	82.7	30.3
0.6	0.0	0.3	3.63	46.9	100	64.8	0.96	42.9	99.9	59.3	4.46

5.2. Findings for a model with two endogenous regressors

As Section 3 on the simulation study of Kiviet and Pleus (2016) shows, designing a just identified simultaneous model with two endogenous variables such that one can easily control the degree of simultaneity, the strength of the instruments and the multicollinearity between the regressors is not self-evident. For the present purpose the situation is even more complex, because we will have to allow for possible invalidity of the instruments as well when analyzing the power of exclusion restriction tests. We proceed as follows.

Let the 5×1 vectors η_i contain (for $i = 1, ..., n$) independent drawings from a five element multivariate standard normal distribution. Now consider the linear transformation

$$d_i = (x_i^{(1)}, x_i^{(2)}, z_i^{(1)}, z_i^{(2)}, u_i)' = A\eta_i, \tag{5.14}$$

with $A = (a_{jl})$ a 5×5 upper-diagonal real valued matrix. To realize that all elements of d_i have unit variance, the five rows of matrix A should all have inner-product unity. This directly implies $a_{55} = 1$ and $u_i = \eta_{i5}$. Note that the final column of A actually represents $(\rho_{x^{(1)}u}, \rho_{x^{(2)}u}, \rho_{z^{(1)}u}, \rho_{z^{(2)}u}, 1)'$. In the simulation we will control these four correlation parameters by choosing empirically relevant values for them, as well as for six other relevant correlations, all in the $(-1, +1)$ interval. The 10 yet unknown elements of A will follow from these these 6+4 correlations, the first four equations of (5.14), which

are

$$x_i^{(1)} = a_{11}\eta_{i1} + a_{12}\eta_{i2} + a_{13}\eta_{i3} + a_{14}\eta_{i4} + \rho_{x^{(1)}u}\eta_{i5} \tag{5.15}$$

$$x_i^{(2)} = a_{22}\eta_{i2} + a_{23}\eta_{i3} + a_{24}\eta_{i4} + \rho_{x^{(2)}u}\eta_{i5} \tag{5.16}$$

$$z_i^{(1)} = a_{33}\eta_{i3} + a_{34}\eta_{i4} + \rho_{z^{(1)}u}\eta_{i5} \tag{5.17}$$

$$z_i^{(2)} = a_{44}\eta_{i4} + \rho_{z^{(2)}u}\eta_{i5}, \tag{5.18}$$

and the imposed unit variance of all elements of d_i. The unit variance of (5.18) implies

$$a_{44} = (1 - \rho^2_{z^{(2)}u})^{1/2}. \tag{5.19}$$

By controlling the value of $\rho_{z^{(1)}z^{(2)}}$, which follows from (5.17) and (5.18) to be $\rho_{z^{(1)}u}\rho_{z^{(2)}u} + a_{34}a_{44}$, we find

$$a_{34} = (\rho_{z^{(1)}z^{(2)}} - \rho_{z^{(1)}u}\rho_{z^{(2)}u})/a_{44}. \tag{5.20}$$

Correlating (5.18) and (5.16) we find $\rho_{z^{(2)}x^{(2)}} = a_{44}a_{24} + \rho_{z^{(2)}u}\rho_{x^{(2)}u}$, so

$$a_{24} = (\rho_{z^{(2)}x^{(2)}} - \rho_{z^{(2)}u}\rho_{x^{(2)}u})/a_{44}, \tag{5.21}$$

and correlating (5.18) and (5.15) gives $\rho_{z^{(2)}x^{(1)}} = a_{44}a_{14} + \rho_{z^{(2)}u}\rho_{x^{(1)}u}$, hence

$$a_{14} = (\rho_{z^{(2)}x^{(1)}} - \rho_{z^{(2)}u}\rho_{x^{(1)}u})/a_{44}. \tag{5.22}$$

Due to the unit variance of (5.17) we have

$$a_{33} = (1 - a_{34}^2 - \rho^2_{z^{(1)}u})^{1/2}. \tag{5.23}$$

Then from $\rho_{z^{(1)}x^{(2)}} = a_{33}a_{23} + a_{34}a_{24} + \rho_{z^{(1)}u}\rho_{x^{(2)}u}$ we obtain

$$a_{23} = (\rho_{z^{(1)}x^{(2)}} - a_{34}a_{24} - \rho_{z^{(1)}u}\rho_{x^{(2)}u})/a_{33}, \tag{5.24}$$

and from $\rho_{z^{(1)}x^{(1)}} = a_{33}a_{13} + a_{34}a_{14} + \rho_{z^{(1)}u}\rho_{x^{(1)}u}$ we find

$$a_{13} = (\rho_{z^{(1)}x^{(1)}} - a_{34}a_{14} - \rho_{z^{(1)}u}\rho_{x^{(1)}u})/a_{33}. \tag{5.25}$$

The unit variance of (5.16) yields

$$a_{22} = (1 - a_{23}^2 - a_{24}^2 - \rho^2_{x^{(2)}u})^{1/2}, \tag{5.26}$$

and from $\rho_{x^{(1)}x^{(2)}} = a_{12}a_{22} + a_{13}a_{23} + a_{14}a_{24} + \rho_{x^{(1)}u}\rho_{x^{(2)}u}$ we get

$$a_{12} = (\rho_{x^{(1)}x^{(2)}} - a_{13}a_{23} - a_{14}a_{24} - \rho_{x^{(1)}u}\rho_{x^{(2)}u})/a_{22}, \tag{5.27}$$

and at long last

$$a_{11} = (1 - a_{12}^2 - a_{13}^2 - a_{14}^2 - \rho^2_{x^{(1)}u})^{1/2}. \tag{5.28}$$

So, all elements of matrix A can be expressed in the 10 correlations $\rho_{x^{(1)}u}$, $\rho_{x^{(2)}u}$, $\rho_{z^{(1)}u}$, $\rho_{z^{(2)}u}$, $\rho_{x^{(1)}x^{(2)}}$, $\rho_{z^{(1)}x^{(1)}}$, $\rho_{z^{(1)}x^{(2)}}$, $\rho_{z^{(2)}x^{(1)}}$, $\rho_{z^{(2)}x^{(2)}}$ and $\rho_{z^{(1)}z^{(2)}}$. Not all combinations of values for these correlations in the $(-1, 1)$ interval will be compatible though. Obvious requirements are

$$\left.\begin{array}{c} a_{34}^2 + \rho^2_{z^{(1)}u} < 1, \\ a_{23}^2 + a_{24}^2 + \rho^2_{x^{(2)}u} < 1, \\ a_{12}^2 + a_{13}^2 + a_{14}^2 + \rho^2_{x^{(1)}u} \leq 1. \end{array}\right\} \tag{5.29}$$

We will examine just a few compatible combinations of the ten correlations which seem relevant and just considered solutions on the basis of the positive square roots for the diagonal elements of A.

That all elements of d_i have unit variance does not lead to loss of generality. The values to be chosen for β_1 and β_2 can compensate for the unit variance of $x_i^{(1)}$ and $x_i^{(2)}$ in the model $y_i = x_i^{(1)}\beta_1 + x_i^{(2)}\beta_2 + u_i$. The KLS based test of joint restrictions on β_1 and β_2 can be shown to be invariant with respect to β_1 and β_2 when the null is true, and so is the KLS based test on the joint significance of $z_i^{(1)}$ and $z_i^{(2)}$ when added to this model, both under the null and alternatives, and it is also invariant to the scale of all regressors. So when investigating the size of the KLS test of joint restrictions on β_1 and β_2 and the rejection probability both under the null and under alternatives for the joint exclusion restrictions test we may without loss of generality set $\beta_1 = \beta_2 = 0$.

In the simulations we will again take all vectors d_i in deviation from sample averages, so the results are actually about models that include an intercept as well, whereas they in fact also hold for models which yield similar vectors d_i after partialling out any number of arbitrary further exogenous regressors. Table 4 presents some results for the two types of KLS tests for the ideal (but unrealistic) situation that $r = \rho_{xu}$ (true value of the degree of simultaneity is known). We examined all 1024 combinations of $\rho_{x^{(1)}u} \in \{0.2, 0.5\}$, $\rho_{x^{(2)}u} \in \{0.0, 0.3\}$, $\rho_{z^{(1)}u} \in \{0.0, 0.4\}$, $\rho_{z^{(2)}u} \in \{0.0, 0.2\}$, $\rho_{x^{(1)}x^{(2)}} \in \{0.2, 0.6\}$, $\rho_{z^{(1)}x^{(1)}} \in \{0.3, 0.6\}$, $\rho_{z^{(1)}x^{(2)}} \in \{0.0, 0.3\}$, $\rho_{z^{(2)}x^{(1)}} \in \{0.1, 0.3\}$, $\rho_{z^{(2)}x^{(2)}} \in \{0.2, 0.5\}$ and $\rho_{z^{(1)}z^{(2)}} \in \{0.0, 0.3\}$, but present only 64 of them in Table 4. The table has two panels. In the left one we present all 32 results for the lower values of $\rho_{z^{(1)}x^{(1)}}$, $\rho_{z^{(1)}x^{(2)}}$, $\rho_{z^{(2)}x^{(1)}}$, $\rho_{z^{(2)}x^{(2)}}$ and $\rho_{z^{(1)}z^{(2)}}$, and in the right-hand panel those for their higher values. In all experiments the chosen correlation coefficients obeyed the compatibility criteria (5.29). R_x represents the rejection frequency (representing the estimated actual significance level) of the joint significance test on $x_i^{(1)}$ and $x_i^{(2)}$. This test existed in all experiments because we always found $\hat{\theta}(r) > 0$. R_z is the rejection frequency of the joint exclusion restriction test on $z_i^{(1)}$ and $z_i^{(2)}$, which represents its estimated actual significance level for cases where $\rho_{z^{(1)}u} = \rho_{z^{(2)}u} = 0$. For R_z results marked with an asteriks we found $\hat{\theta}(r) \leq 0$ in less than 0.1% of the replications. However, in a few of the cases not included in the table we found the exclusion test to be undefined much more frequently.

From Table 4 we observe that the size properties of both unfeasible KLS test procedures are also very reasonable in the more general model, whereas the power of the exclusion restrictions test seems fine. This being the case for the ideal situation in which $\rho_{x^{(1)}u}$ and $\rho_{x^{(2)}u}$ are supposed to be known provides the appropriate starting point for reasonably successful implementations under more realistic assumptions, as we saw in the preceding subsection.

Table 4: Rejection frequencies (in %) of unfeasible KLS tests ($K_1 = L_2 = 2; n = 100; \alpha = 0.05$)

$\rho_{z(1)z(2)} = 0.0$; $\rho_{z(2)x(2)} = 0.2$; $\rho_{z(2)x(1)} = 0.1$; $\rho_{z(1)x(2)} = 0.0$; $\rho_{z(1)x(1)} = 0.3$;							$\rho_{z(1)z(2)} = 0.3$; $\rho_{z(2)x(2)} = 0.5$; $\rho_{z(2)x(1)} = 0.3$; $\rho_{z(1)x(2)} = 0.3$; $\rho_{z(1)x(1)} = 0.6$;						
$\rho_{x(1)x(2)}$	$\rho_{z(2)u}$	$\rho_{z(1)u}$	$\rho_{x(2)u}$	$\rho_{x(1)u}$	R_x	R_z	$\rho_{x(1)x(2)}$	$\rho_{z(2)u}$	$\rho_{z(1)u}$	$\rho_{x(2)u}$	$\rho_{x(1)u}$	R_x	R_z
0.2	0.0	0.0	0.0	0.2	5.13	5.15	0.2	0.0	0.0	0.0	0.2	5.14	5.15
				0.5	5.03	4.73					0.5	5.03	4.12
			0.3	0.2	5.18	5.09				0.3	0.2	5.09	4.89
				0.5	5.07	4.65					0.5	5.00	4.05*
		0.4	0.0	0.2	5.12	97.7			0.4	0.0	0.2	5.15	99.8
				0.5	5.03	96.9					0.5	5.02	99.2
			0.3	0.2	5.16	97.9				0.3	0.2	5.08	99.8
				0.5	5.06	97.1					0.5	5.01	97.4*
	0.2	0.0	0.0	0.2	5.14	42.9		0.2	0.0	0.0	0.2	5.15	55.0
				0.5	5.04	42.1					0.5	5.02	51.1
			0.3	0.2	5.18	41.7				0.3	0.2	5.10	50.3
				0.5	5.07	40.3					0.5	5.02	36.0*
		0.4	0.0	0.2	5.14	99.5			0.4	0.0	0.2	5.16	100
				0.5	5.03	99.3					0.5	5.04	99.8
			0.3	0.2	5.16	99.5				0.3	0.2	5.09	99.9
				0.5	5.07	99.2					0.5	5.05	97.3*
0.6	0.0	0.0	0.0	0.2	5.14	5.11	0.6	0.0	0.0	0.0	0.2	5.11	5.11
				0.5	4.72	3.81					0.5	4.74	4.02*
			0.3	0.2	5.20	5.09				0.3	0.2	5.10	5.06
				0.5	5.09	4.71					0.5	5.06	4.11
		0.4	0.0	0.2	5.14	98.1			0.4	0.0	0.2	5.12	99.8
				0.5	4.71	96.0					0.5	4.71	91.6*
			0.3	0.2	5.17	98.7				0.3	0.2	5.11	99.9
				0.5	5.12	97.5					0.5	5.06	99.2
	0.2	0.0	0.0	0.2	5.15	43.1		0.2	0.0	0.0	0.2	5.11	56.5
				0.5	4.74	43.0					0.5	4.73	59.9*
			0.3	0.2	5.19	41.8				0.3	0.2	5.11	51.4
				0.5	5.10	41.6					0.5	5.07	50.7
		0.4	0.0	0.2	5.15	99.5			0.4	0.0	0.2	5.11	99.9
				0.5	4.72	99.0					0.5	4.72	95.2*
			0.3	0.2	5.18	99.7				0.3	0.2	5.12	99.8
				0.5	5.13	99.3					0.5	5.05	99.1

6. Empirical illustrations

We will produce empirical results for three different cross-section data sets.

6.1. A wage equation for employed women

As a first illustration we closely follow a textbook example on IV/TSLS estimation given in Carter Hill et al. (2012, p.415). It concerns a subset of data originating from Mroz, namely a few variables on a sample of $n = 428$ employed women. After taking all these variables in deviation from their mean, the relationship considered is a special case of model (2.1), namely

$$y_i = \beta_1 x_i^{(1)} + x_i^{(2)\prime}\beta_2 + u_i, \tag{6.1}$$

with $K_1 = 1$ and $K_2 = 2$, where y_i is the log of wage, $x_i^{(1)}$ is education in years and vector $x_i^{(2)}$ contains the variables experience in years and its square. It is assumed that $x_i^{(1)}$ is actually a proxy for the unavailable variable $\mathring{x}_i^{(1)}$, which should express ability or intelligence. So, implicitly it is assumed that applying OLS to the model

$$y_i = \mathring{\beta}_1 \mathring{x}_i^{(1)} + x_i^{(2)\prime}\mathring{\beta}_2 + \mathring{u}_i \tag{6.2}$$

would yield consistent estimates, hence $E(\mathring{x}_i^{(1)}\mathring{u}_i) = 0$ and $E(x_i^{(2)}\mathring{u}_i) = 0$.

However, regression (6.2) being unfeasible, the use in model (6.1) of $L_2 = 2$ external instrumental variables $z_i^{(2)} = (z_{i1}^{(2)}, z_{i2}^{(2)})$ is being considered, where $z_{i1}^{(2)}$ is the education in years of the mother and $z_{i2}^{(2)}$ of the father of the woman concerned. The two variables in $x_i^{(2)}$ are used as internal instruments, which requires $E(x_i^{(2)}u_i) = 0$, next to $E(z_i^{(2)}u_i) = 0$.

It does not seem unreasonable to assume that the proxy variable $x_i^{(1)}$ can be decomposed as

$$x_i^{(1)} = \alpha_1 \mathring{x}_i^{(1)} + x_i^{(2)\prime}\alpha_2 + v_i, \tag{6.3}$$

where $\alpha_1 > 0$, and remainder v_i is such that $E(\mathring{x}_i^{(1)} v_i) = 0$ and $E(x_i^{(2)} v_i) = 0$. Then substituting (6.3) into (6.2) gives

$$\begin{aligned} y_i &= \mathring{\beta}_1 \alpha_1^{-1}(x_i^{(1)} - x_i^{(2)\prime}\alpha_2 - v_i) + x_i^{(2)\prime}\mathring{\beta}_2 + \mathring{u}_i \\ &= \mathring{\beta}_1 \alpha_1^{-1} x_i^{(1)} + x_i^{(2)\prime}(\mathring{\beta}_2 - \mathring{\beta}_1 \alpha_1^{-1}\alpha_2) + \mathring{u}_i - \mathring{\beta}_1 \alpha_1^{-1} v_i, \end{aligned}$$

which implies for model (6.1) that $\beta_1 = \mathring{\beta}_1 \alpha_1^{-1}$, $\beta_2 = \mathring{\beta}_2 - \mathring{\beta}_1 \alpha_1^{-1}\alpha_2$ and $u_i = \mathring{u}_i - \mathring{\beta}_1 \alpha_1^{-1} v_i$. From this we find

$$\begin{aligned} E(x_i^{(1)} u_i) &= E[(\alpha_1 \mathring{x}_i^{(1)} + x_i^{(2)\prime}\alpha_2 + v_i)(\mathring{u}_i - \mathring{\beta}_1 \alpha_1^{-1} v_i)] \\ &= E(v_i \mathring{u}_i) - \beta_1 E(v_i^2). \end{aligned}$$

It seems plausible to assume $E(v_i \mathring{u}_i) = 0$, because otherwise $x_i^{(1)}$ would in fact be an omitted variable from regression (6.2). So, we find

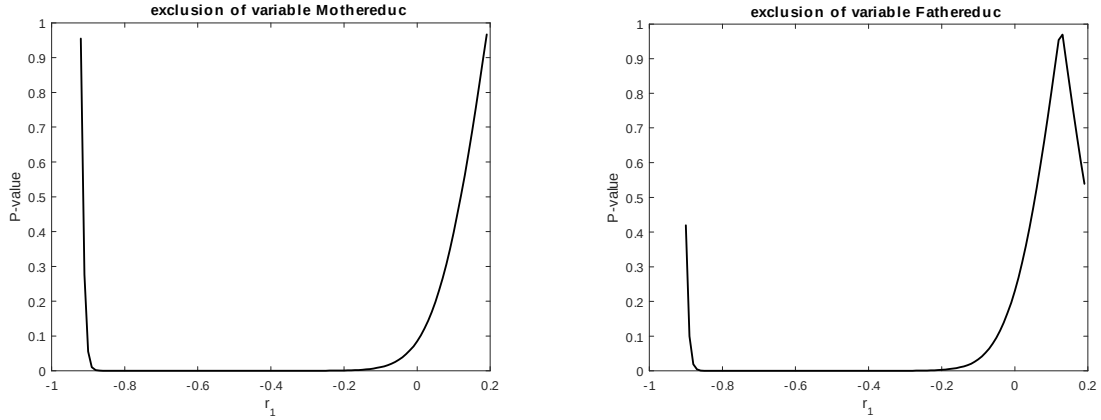
$$\rho_1 = \frac{E(x_i^{(1)} u_i)}{\sigma_1 \sigma_u} = -\beta_1 \frac{E(v_i^2)}{\sigma_1 \sigma_u} = -\delta \frac{\beta_1 \sigma_1}{\sigma_u}, \tag{6.4}$$

assuming that $E(v_i^2) = \delta \sigma_1^2$, where, say, $0.1 < \delta < 0.4$. Coefficient $100 \times \beta_1$ represents the percentage wage increase per extra year of education. Since we expect $0 < \beta_1 < 0.1$, we may suppose $-1 < \rho_1 < 0$, so that the OLS estimate of β_1 will have a negative bias. If $\beta_1 \sigma_1 / \sigma_u$ is about 2 (which it would be on the basis of the TSLS findings), then we should have a special interest in examining the range $-0.8 < \rho_1 < -0.2$.

Because we deduced that $E(x_i^{(1)} u_i) \neq 0$, variable $x_i^{(1)}$ is endogenous in regression (2.1). Its endogeneity is not due to classic simultaneity or dual or reciprocal causality, but it simply stems from an omitted explanatory variable which has been replaced by a proxy variable, so the origin of the endogeneity is actually measurement error. Thus, the endogeneity of $x_i^{(1)}$ is not intrinsic here, but incurred. That $x_i^{(1)}$ is the one and only endogenous regressor in (6.1) is due to the untested assumption $E(x_i^{(2)} \mathring{u}_i) = 0$, which yields $E(x_i^{(2)} u_i) = E[x_i^{(2)}(\mathring{u}_i - \mathring{\beta}_1 \alpha_1^{-1} v_i)] = -\beta_1 E(x_i^{(2)} v_i) = 0$.

Figure 1 shows over a wide range of r_1 values the P-values of the single just-identifying exclusion restriction tests for the variables $z_i^{(1)}$ and $z_i^{(2)}$ respectively. Over the range $-0.9 < \rho_1 < -0.1$ (which we suppose to contain the true value of ρ_1) for both variables all calculated P-values are below 5%. Testing their joint exclusion (results not presented) leads to the same conclusion. Thus, overwhelming evidence has been found forcing to conclude that these instruments are invalid. Nevertheless, the standard methods strongly support the TSLS results for the chosen specification. In reduced form regressions for $x_i^{(1)}$, where next to $x_i^{(2)}$ just $z_{i1}^{(2)}$ is added, its F-test value is 73.95, whereas this is 87.74 for $z_{i2}^{(2)}$, so both instruments seem pretty strong. Jointly they have an F-value of 55.40. Also, the Sargan test for the single over-identification restriction when using both instruments has P-value 0.54. So, according to standard practice methods, acceptance of the TSLS results seems vindicated; the invalidity of the instruments remains undetected.

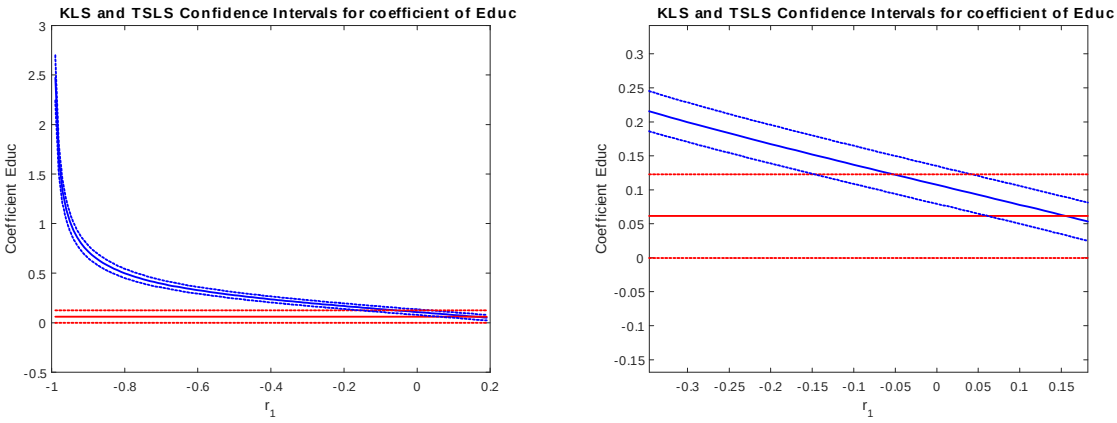
Figure 1: P-values of single just-identifying exclusion restriction tests



There is an aspect overlooked by Carter Hill et al. (2014), which seems to reveal the inconsistency of TSLS for the present model. They find a larger estimate of β_1 by OLS than by TSLS and argue that this is to be expected if variable $x_i^{(1)}$ is positively correlated with the omitted factors in the error term. However, as we demonstrate above, we expect ρ_1 to be negative, and hence the inconsistency of the OLS estimator of β_1 should be negative too. So, supposing TSLS to be consistent, one should expect OLS to yield a smaller estimate than TSLS. The negative sign of ρ_1 implies that the KLS estimator of β_1 will turn out to be larger than the OLS estimator.

Figure 2 shows the TSLS asymptotic 95% confidence interval for β_1 (red dotted lines), which is invariant regarding ρ_1, and is centered at the TSLS estimate 0.0614 (red line). It also shows the KLS estimator (blue line), which varies with r_1 and the KLS asymptotic 95% confidence interval (blue dotted lines). The right-hand-side graph zooms in on the area which we suppose to comprise the true value of β_1. The standard OLS nominal 95% confidence interval is indicated at $r_1 = 0$, centered around 0.108. Figure 2 shows that for substantially negative values of ρ_1 the consistent (when the assumptions of Section 2 apply) KLS estimators produce ludicrous values for β_1. Hence, the conclusion must not simply be that the two external instruments are invalid for model (6.1), as Figure 1 shows, but that more serious specification problems undermine this model than just endogeneity of $x_i^{(1)}$. These problems do not emerge from a standard TSLS analysis, but show up by applying KLS.

Figure 2: Inference on β_1 based on (non-)orthogonality conditions



6.2. An analysis of the weight of newborns

We shall present another simple illustration. In Wooldridge (2010, p.116) an exercise is presented in which it is analyzed for $n = 1388$ newborns whether smoking by their mother during pregnancy affects birth weight. A model like (6.1) is analyzed with $K_1 = 1$ but now $K_2 = 3$, where y_i is the log of birth weight, $x_i^{(1)}$ is the average number of packs of cigarettes smoked during pregnancy and vector $x_i^{(2)}$ contains a dummy for the gender of the baby, a variable parity, which is the birth order of the child, and the log of family income. It is assumed that $x_i^{(1)}$ is correlated with the disturbance term, because various further determinants of birth weight may be correlated with smoking behavior, such as alcohol use, health consciousness, fitness activity, stress, food, sleep and many more, and these have all been omitted from the model. It is suggested to use the price of cigarettes as an instrumental variable, because economic theory predicts that it is negatively correlated with packs smoked, whereas it does not seem likely that this price has a direct effect on birth weight.

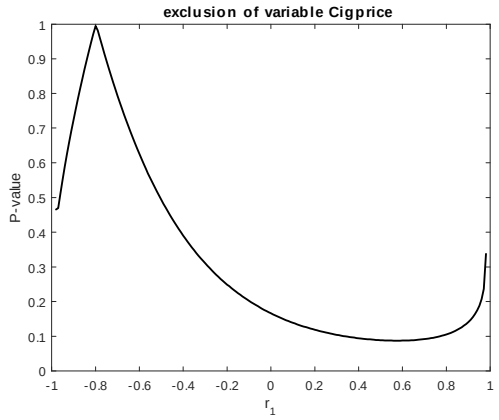
OLS yields a coefficient estimate for packs of -0.084 with standard error 0.017, which (when consistent) would suggest that each extra cigarette smoked per day (each package containing 20 cigarettes) reduces birth weight by about 0.4%. TSLS yields an outrageous coefficient of 0.651 (positive!) with standard error 0.854. These are clearly affected by weakness of the instrument (price elasticity will be moderate because smoking is addictive) since the relevant F-value in the first-stage regression is only 1.00.

To this standard evidence the procedures developed in this study can add the following. The analysis presented at the start of this section can now be interpreted as follows. Suppose $x_i^{(1)}$ is used here as a proxy for the comprehensive latent variable "life-style risks for baby's birth weight". Then we have again $\alpha_1 > 0$, but expect now $\beta_1 < 0$. So here $\rho_1 > 0$, which would render the OLS estimator of β_1 positively biased which suggests that an extra cigarette per day may reduce birth weight by more than 0.4%. Suppose that in fact $-0.08 > \beta_1 > -0.15$, $0.2 < \delta < 0.8$ and $0.5 < \sigma_1/\sigma_u < 2$ then it follows from (6.4) that $0.008 < \rho_1 < 0.24$. From the left-hand side of Figure 3 we can see that over this area the validity of the instrument lacks strong support. Although the exclusion test does not force to reject at a significance level smaller than 10%, in order to justify the use of the instrument a P-value much larger, say exceeding 50%, would provide much more comfort. The relatively low P-values for $\rho_1 > 0$ also do not encourage to move on to applying weak instrument techniques.

Assuming that the conditions to apply KLS do hold, the right-hand-side graph in Figure 3 shows that if we knew ρ_1 we would be able to produce highly accurate KLS inference on β_1 (blue lines; the confidence interval is so narrow that the figure barely shows it). For instance, it enables to infer rejection of the hypothesis $\beta_1 > 0$, provided $\rho_1 > -0.05$. KLS also allows a sensitivity analysis of TSLS: It shows that the extremely wide (and hence pretty useless) TSLS confidence interval is conservative at the nominal 95% level, provided $-0.95 < \rho_1 < 0.9$. Zooming in on this figure yields the left-hand side of Figure 4, from which we can deduce that for $0 \leq \rho_1 \leq 0.35$ (which does not seem unrealistic) the confidence set $-0.36 \leq \beta_1 \leq -0.05$ has asymptotic confidence coefficient 0.95. In the right-hand side of Figure 4 we produce KLS inference on one of the coefficients of the exogenous regressors, namely the log of family income. The TSLS estimate of this coefficient is 0.064 and its 95% confidence interval is $(-0.048, 0.175)$, but KLS learns that for realistic values of ρ_1 this coefficient is much smaller and significantly

negative. Hence, for realistic values of ρ_1 the TSLS interval is liberal here (actual confidence coefficient smaller than the nominal coefficient), whereas it is predominantly conservative in the right-hand side of Figure 3.

Figure 3: Inference on birth weight data based on (non-)orthogonality conditions



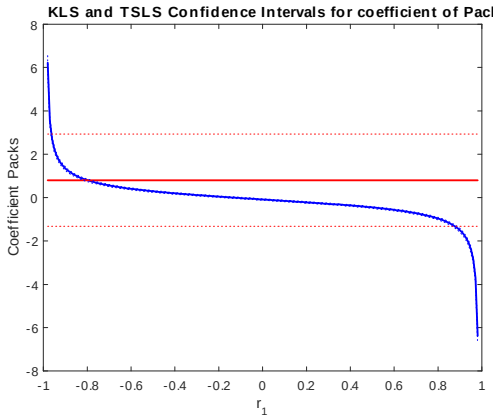
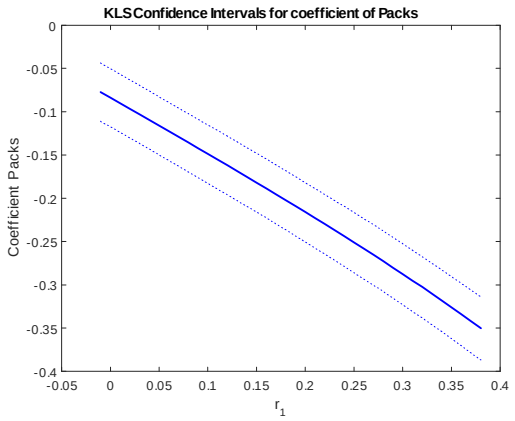
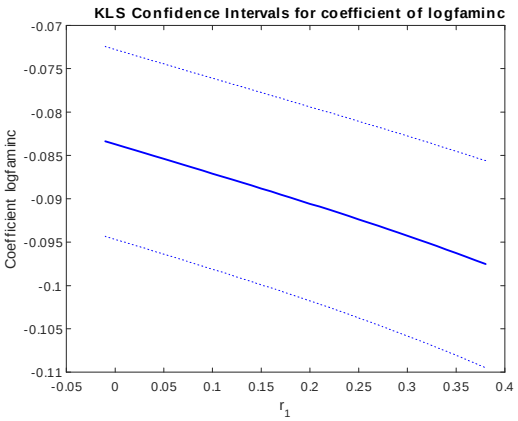


Figure 4: KLS inference on coefficients of birth weight data



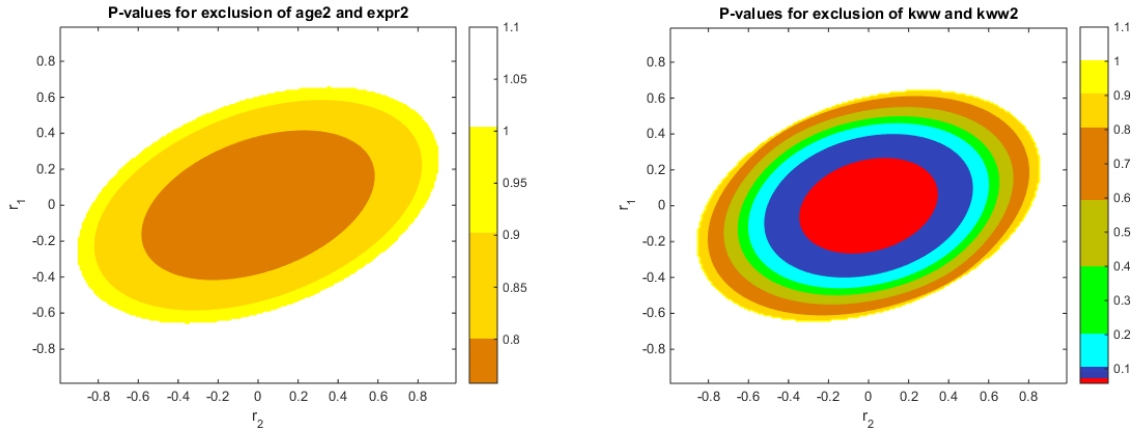


6.3. A wage equation for young men

The above illustrations required using the simple Corollary 1.2 only, because they concern models with just one endogenous regressor. Next we will exploit Theorem 1 in its full complexity in an empirical model where $K_1 = 2$ which is based on a classic data set originating from work by Griliches and also used for illustrative purposes on a subset ($n = 758$) of these data on young men in Hayashi (2000, p.251). In Kiviet and Pleus (2017, p.18) we used the same data to illustrate tests on establishing the endogeneity of subsets of regressors. These are built on assuming validity of two untestable (according to the classic approach) identifying orthogonality conditions. Like in the first illustration log wage is the dependent variable, but next to schooling also an iq test score is a possibly endogenous regressor in addition to a range of exogenous controls ($K_2 = 11$), including age and experience. The external instruments used are ($L_2 = 4$): age2 (age squared), expr2 (experience squared), kww (another test score) and kww2 (kww squared). The overall Sargan test (2 degrees of freedom) has satisfying P-value 0.89, but it leaves two underlying just-identifying restrictions untested.

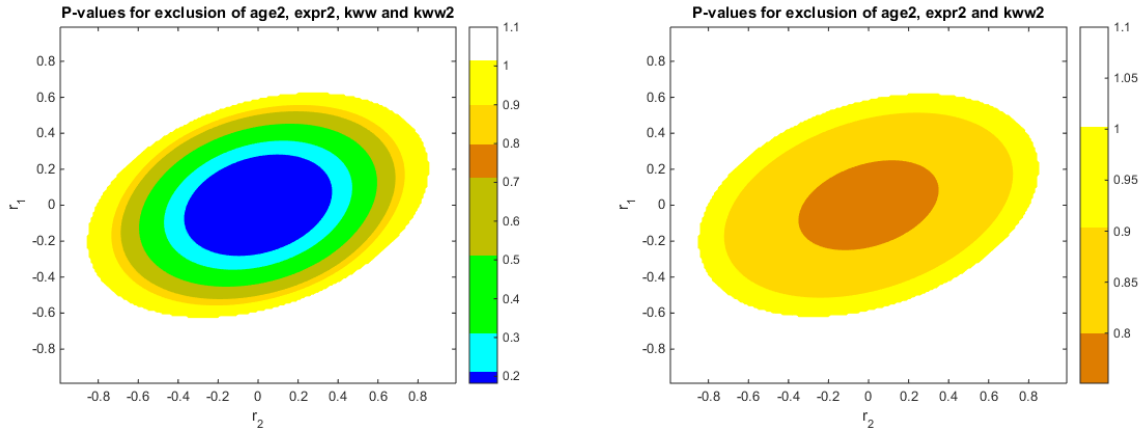
Figure 5 shows (colored) contour plots for the P-values of the KLS based exclusion restrictions tests ($L_2 = 2$) of age2 and expr2 (left-hand) and kww and kww2 (right-hand) respectively. These plots have been obtained by calculating test statistic W^* of (4.9) over a range of values for the simultaneity correlations, where r_1 refers to schooling and r_2 to iq score. For both the grid values -0.99:0.01:0.99 have been examined. For cases where $r'S_xS_{xx}^{-1}S_xr > 0.99$ we did set the P-value at 1.1, to be interpreted as "not defined". Both plots show that the statistic is defined over an ellipse. The exclusion restriction regarding the squares of both included regressors age and expr does not have to be rejected whatever the true values of ρ_1 and ρ_2 will be, since all P-values exceed 0.75. This is pretty hard evidence (although not irrefutable) on the possible validity of these two instruments. For score test variable kww and its square the situation is different. Over a substantial area of (ρ_1, ρ_2) combinations their exclusion test has P-values well below 0.1, whereas the area where it exceeds 0.7 forms just a narrow shell, covering cases where $\rho_1^2 + \rho_2^2$ is relatively large. Especially when the simultaneity is nonexistent or mild the validity of kww and kww2 as instruments seems doubtful. Assuming that both schooling and iq are positively related to "ability", we expect both ρ_1 and ρ_2 to be mildly positive.

Figure 5: P-values of two just-identifying exclusion restriction tests



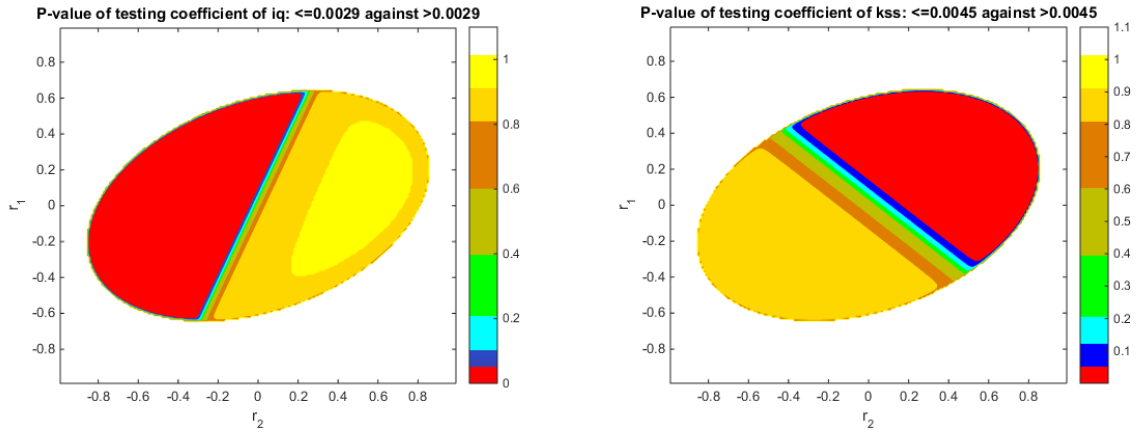
In the left-hand side contour plot of Figure 6 we test the exclusion of the $L_2 = 4$ variables jointly. Now no P-values are obtained below 0.18. This demonstrates that the exclusion test may have limited power when some rightly (age2 and expr2) and some wrongly (kww and kww2) excluded regressors are tested jointly. In the right-hand side of Figure 6 we test the model in which kww has been included as an exogenous regressor ($K_2 = 12$), and it is tested whether the $L_2 = 3$ squared variables seem valid external instruments for the two endogenous regressors schooling and iq. Figure 6 highlights that the inference on endogeneity of these two regressors as presented in Kiviet and Pleus (2017), which uses the model and instruments of the left-hand contour plot, although supported by a large P-value of the Sargan test, should better have been executed in the model and with the external instruments of the right-hand contour plot, because this does not discourage the use of these three instruments irrespective of the actual values of ρ_1 and ρ_2.

Figure 6: P-values of exclusion restriction tests in two different models



The results just found can either be used to precede a traditional TSLS analysis, as performed in Kiviet and Pleus (2017). Or it can be interpreted as supporting the specification of the model which includes kww as an extra regressor and excludes the three squared variables ($K_2 = 12$), while treating both schooling and iq as endogenous ($K_1 = 2$), and next –avoiding the use of possibly weak or invalid instruments– analyze its coefficients on the basis of KLS inference. We will do the latter here, just focussing on the coefficients of iq and kss. We perform one sided tests on the single hypothesis that these coefficients exceed some particular value. For this value we chose their estimated values as obtained by OLS, which are 0.0029 and 0.0045 respectively. From the contours of Figure 7 one can see that, roughly, this hypothesis is rejected for iq when $\rho_1 > 2\rho_2$ and for kss when $\rho_1 > -0.9\rho_2$. Assuming both ρ_1 and ρ_2 to be about 0.3 it seems likely that the coefficient of iq is smaller than 0.0029 and that of kss larger than 0.0045.

Figure 7: KLS inference on particular coefficients



To depict similar inference results when we would allow kss to be endogenous too becomes of course much more complicated. However, it is not impossible. One could produce, next to the above results where kss is supposed to be exogenous (which could be expressed as $\rho_3 = 0$), also contours for some cases where, for instance, $\rho_3 = 0.1 : 0.1 : 0.5$. Hence, such contour plots allow to examine the sensitivity of OLS or TSLS coefficient estimates, but also to produce inference on either instrument validity or regression coefficients conditional on specific assumptions regarding simultaneity or to some degree robust to simultaneity.

7. Conclusion

By incremental or difference Sargan-Hansen tests for over-identifying restrictions the validity of a subgroup of instruments can be tested, provided a sufficient number of valid (i.e. exogenous) and relevant (i.e. sufficiently strong) over- or just-identifying instruments are already available. When one wants to verify whether these latter instruments are really valid indeed, the only route provided by the standard approach is: first adopt another non-testable set of valid identifying instruments. Thus, providing statistical evidence on the validity of all instruments is simply impossible by these tools. It mimics a situation where for a proof by mathematical induction one can prove the induction step, whereas proof of the truth of a base case is yet missing, and its proof seems even completely beyond reach.

However, a particular implementation of the KLS-based test procedure developed in this study, by which general linear restrictions can be tested in a multiple regression model with an arbitrary number of endogenous regressors without exploiting any external instruments –which is of substantial practical relevance by itself– also allows to generate statistical evidence on the tenability of exclusion restrictions. In situations where in this way a just-identifying or over-identifying set of acceptable instruments has been established, it provides for a series of incremental Sargan-Hansen tests the essential underlying building block which was missing so far. This supporting building block seems mandatory if one still wants to employ IV-based estimation and inference. However, the general tools developed in this paper also allow to produce inference on coefficient values while avoiding the use of external instruments altogether and thus missing out all the ensuing problems such as sacrificing credibility, accuracy and power due to possible weakness or invalidity of instruments.

The tools developed here are not very demanding computationally, and can also be used to provide a sensitivity analysis of least-squares or instrumental variables based inferences with respect to less strict assumptions regarding the orthogonality assumptions on which these are built.

Of course, as always, deeper insights and further generalizations are called for. Preceding the usual Sargan-Hansen tests by a just-identifying exclusion restrictions test exacerbates the pre-test problems. Theorem 1 presupposes homoskedasticity of both disturbances and regressors, so if this is not the case one should manage to first weigh all observations such that this is achieved as closely as possible. Developing inference methods which are robust regarding both simultaneity and heteroskedasticity and at the same time control size and boost power over the whole model building process remain a challenge for the future efforts.

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Appendices

A. Some basic derivations

We have assumed that $\{(x_i', u_i)'; i = 1, ..., n\}$ are independently and identically distributed with zero mean and

$$Var \begin{pmatrix} x_i \\ u_i \end{pmatrix} = \begin{pmatrix} \Sigma_{xx} & \sigma_{xu} \\ \sigma_{xu}' & \sigma_u^2 \end{pmatrix}.$$

All elements of the latter matrix are assumed to be finite. In addition, we assume that all elements of $(x_i', u_i)'$ have a symmetric distribution, whereas $E(u_i^4) = \kappa_u \sigma_u^4$ and $E(x_{ij}^4) = \kappa_x \sigma_j^4$, with $1 \leq \kappa_u < \infty$ and $1 \leq \kappa_x < \infty$. We denote the typical element of Σ_{xx} by σ_{jk} $(j, k = 1, ..., K)$, but for its diagonal elements we will sometimes use $\sigma_j^2 = \sigma_{jj}$. The typical element of vector σ_{xu} can be denoted $\sigma_u \sigma_j \rho_j$, because $\rho_j = \sigma_{x_j u}/(\sigma_j \sigma_u)$. Next to Σ_{xx} and its sample equivalent S_{xx}, where for the latter we defined two options at the end of Section 2, we will also use the matrices $\Sigma_x^2 = \text{diag}(\sigma_1^2, ..., \sigma_K^2)$ and $\Sigma_x = \text{diag}(\sigma_1, ..., \sigma_K)$, as well as the diagonal matrices S_x and S_x^2. The latter has the same main diagonal as S_{xx}, and $S_x S_x = S_x^2$.

Invoking a standard version of the central limit theorem, we can now obtain the following results, which will be exploited later. We have

$$n^{1/2}(u'u/n - \sigma_u^2) = n^{-1/2}\Sigma_{i=1}^n (u_i^2 - \sigma_u^2) \xrightarrow{d} \mathcal{N}[0, (\kappa_u - 1)\sigma_u^4], \tag{A.1}$$

because $Var(u_i^2 - \sigma_u^2) = E(u_i^4) - \sigma_u^4 = (\kappa_u - 1)\sigma_u^4$. So, $u'u/n - \sigma_u^2 = O_p(n^{-1/2})$. Also

$$n^{1/2}(X'u/n - \sigma_{xu}) = n^{-1/2}\Sigma_{i=1}^n (x_i u_i - \sigma_{xu}) \xrightarrow{d} \mathcal{N}[0, \sigma_u^2 \Sigma_{xx} + (\kappa_u - 2)\sigma_{xu}\sigma_{xu}'], \tag{A.2}$$

hence $X'u/n - \sigma_{xu} = O_p(n^{-1/2})$. This is found by decomposing x_i into two components, $x_i = \xi_i + \sigma_{xu}\sigma_u^{-2} u_i$, where ξ_i is independent of u_i. Of course, $E(\xi_i) = 0$ and $E(\xi_i u_i) = 0$, so $E(x_i u_i) = \sigma_{xu}$ indeed. Since $Var(x_i) = \Sigma_{xx} = Var(\xi_i) + \sigma_u^{-2}\sigma_{xu}\sigma_{xu}'$, result (A.2) follows from $Var(x_i u_i - \sigma_{xu}) = E(u_i^2 x_i x_i') - \sigma_{xu}\sigma_{xu}' = E(u_i^2 \xi_i \xi_i') + \sigma_{xu}\sigma_{xu}'\sigma_u^{-4} E(u_i^4) - \sigma_{xu}\sigma_{xu}' = \sigma_u^2 Var(\xi_i) + (\kappa_u - 1)\sigma_{xu}\sigma_{xu}' = \sigma_u^2 \Sigma_{xx} + (\kappa_u - 2)\sigma_{xu}\sigma_{xu}'$.

For $j, k = 1, ..., K$ we have

$$n^{1/2}(X'X/n - \Sigma_{xx})_{j,k} = n^{-1/2}\Sigma_{i=1}^n (x_{ij}x_{ik} - \sigma_{jk}) \xrightarrow{d} \mathcal{N}[0, \sigma_j^2\sigma_k^2 + (\kappa_x - 2)\sigma_{jk}^2]. \tag{A.3}$$

This is proved by decomposing $x_{ik} = a_1 x_{ij} + a_2 \eta_{ik}$, where x_{ij} and η_{ik} are independent and η_{ik} has zero mean and unit variance. Because $E(x_{ik}^2) = \sigma_k^2 = a_1^2\sigma_j^2 + a_2^2$ and $E(x_{ik}x_{ij}) = \sigma_{kj} = a_1\sigma_j^2$ we have $a_1 = \sigma_{kj}\sigma_j^{-2}$ and $a_2^2 = \sigma_k^2 - \sigma_{kj}^2\sigma_j^{-2}$. Now we obtain $E(x_{ij}^2 x_{ik}^2) = E[x_{ij}^2(a_1^2 x_{ij}^2 + 2a_1 a_2 x_{ij}\eta_{ik} + a_2^2\eta_{ik}^2)] = \kappa_x\sigma_{jk}^2 + \sigma_j^2(\sigma_k^2 - \sigma_{kj}^2\sigma_j^{-2}) = \sigma_j^2\sigma_k^2 + (\kappa_x - 1)\sigma_{jk}^2$, thus $Var(x_{ij}x_{ik} - \sigma_{jk}) = \sigma_j^2\sigma_k^2 + (\kappa_x - 2)\sigma_{jk}^2$, from which (A.3) follows. So, $n^{-1}X'X - \Sigma_{xx} = O_p(n^{-1/2})$.

Another result that we will exploit later, which involves the Hadamard (element by element) matrix product (denoted $\circ$), is

$$n^{1/2}(S_x^2 - \Sigma_x^2)\rho_{xu} \xrightarrow{d} \mathcal{N}[0, (\kappa_x - 1)\mathcal{R}(\Sigma_{xx} \circ \Sigma_{xx})\mathcal{R}], \tag{A.4}$$

where $\mathcal{R} = \text{diag}(\rho_1, ..., \rho_K)$. Since, following the first option for S_x^2, we have $n^{1/2}(S_x^2 - \Sigma_x^2)\rho_{xu} = n^{-1/2}\Sigma_{i=1}^n v_i$ with $v_i' = ((x_{i1}^2 - \sigma_1^2)\rho_1, ..., (x_{iK}^2 - \sigma_K^2)\rho_K)'$, using the expression for $E(x_{ij}^2 x_{ik}^2)$ just derived we find $E[(x_{ij}^2 - \sigma_j^2)(x_{ik}^2 - \sigma_k^2)] = (\kappa_x - 1)\sigma_{jk}^2$. Thus $E[(x_{ij}^2 - \sigma_j^2)\rho_j(x_{ik}^2 - \sigma_k^2)\rho_k] = (\kappa_x - 1)\rho_j\sigma_{jk}^2\rho_k$, which is the typical element of the limiting variance matrix of (A.4).

In what follows we also need the mutual covariances of scalar (A.1) and vectors (A.2) and (A.4). We find $E[(u_i^2 - \sigma_u^2)(x_i u_i - \sigma_{xu})] = E[(u_i^2 - \sigma_u^2)(\xi_i u_i + \sigma_{xu}\sigma_u^{-2}u_i^2 - \sigma_{xu})] = (\kappa_u - 1)\sigma_u^2\sigma_{xu}$, hence

$$nE[(u'u/n - \sigma_u^2)(X'u/n - \sigma_{xu})] = (\kappa_u - 1)\sigma_u^2\sigma_{xu}. \tag{A.5}$$

Using $x_{ij} = \xi_{ij} + \rho_j\sigma_j\sigma_u^{-1}u_i$, from $E[(u_i^2 - \sigma_u^2)(x_{ij}^2 - \sigma_j^2)\rho_j] = \rho_j E[(u_i^2 - \sigma_u^2)(\xi_{ij}^2 + 2\sigma_j\sigma_u^{-1}u_i\xi_{ij} + \rho_j^2\sigma_j^2\sigma_u^{-2}u_i^2)] = (\kappa_u - 1)\sigma_u^2\rho_j^3\sigma_j^2$ we find

$$nE[(u'u/n - \sigma_u^2)(S_x^2 - \Sigma_x^2)\rho_{xu}] = (\kappa_u - 1)\sigma_u^2\Sigma_x^2\mathcal{R}^2\rho_{xu}. \tag{A.6}$$

And, using $Var(\xi_i) = \Sigma_{xx} - \sigma_u^{-2}\sigma_{xu}\sigma_{xu}'$, from

$$\begin{aligned}
&E[(\xi_{ij}u_i + \rho_j\sigma_j\sigma_u^{-1}u_i^2 - \rho_j\sigma_j\sigma_u)(\xi_{ik}^2 + 2\rho_k\sigma_k\sigma_u^{-1}\xi_{ik}u_i + \rho_k^2\sigma_k^2\sigma_u^{-2}u_i^2 - \sigma_k^2)\rho_k] \\
&= 2\rho_k^2\sigma_k\sigma_u(\Sigma_{xx} - \sigma_u^{-2}\sigma_{xu}\sigma_{xu}')_{jk} + (\kappa_u - 1)\rho_j\sigma_j\sigma_u\rho_k^3\sigma_k^2 \\
&= 2\sigma_u\rho_k^2\sigma_k(\Sigma_{xx})_{jk} + (\kappa_u - 3)\sigma_u\sigma_j\rho_j\rho_k^3\sigma_k^2
\end{aligned}$$

we obtain

$$nE[(X'u/n - \sigma_{xu})\rho_{xu}'(S_x^2 - \Sigma_x^2)] = 2\sigma_u\Sigma_{xx}\Sigma_x\mathcal{R}^2 + (\kappa_u - 3)\sigma_u\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x^2\mathcal{R}^2. \tag{A.7}$$

B. Proof of Theorem 1

To find the limiting distribution of the inconsistency corrected OLS estimator $\hat{\beta}(\rho_{xu}) = \hat{\beta}_{OLS} - n \cdot \hat{\sigma}_u(\rho_{xu})(X'X)^{-1}S_x\rho_{xu}$ we examine

$$n^{1/2}[\hat{\beta}(\rho_{xu}) - \beta] = (n^{-1}X'X)^{-1}[n^{-1/2}X'u - n^{1/2}\hat{\sigma}_u(\rho_{xu})S_x\rho_{xu}]. \tag{B.1}$$

First, we have to separate from the right-hand side expression the leading $O_p(1)$ terms from $o_p(1)$ terms. Matrix $n^{-1}X'X = O_p(1)$ can be decomposed as

$$n^{-1}X'X = \Sigma_{xx} + (n^{-1}X'X - \Sigma_{xx}), \tag{B.2}$$

where the first component is deterministic and finite, denoted as $\Sigma_{xx} = O(1)$, and the second component is $n^{-1}X'X - \Sigma_{xx} = O_p(n^{-1/2})$, see derivation below (A.3). Exploiting the smaller order of this second component we find

$$\begin{aligned}
(n^{-1}X'X)^{-1} &= (\Sigma_{xx} + n^{-1}X'X - \Sigma_{xx})^{-1} = \Sigma_{xx}^{-1}[I + (n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1}]^{-1} \\
&= \Sigma_{xx}^{-1}[I - (n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1} \\
&\qquad + (n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1}(n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1} - ...] \\
&= \Sigma_{xx}^{-1} - \Sigma_{xx}^{-1}(n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1} + o_p(n^{-1/2}).
\end{aligned} \tag{B.3}$$

Hence, this inverse has a leading $O(1)$ term, a second term of order $O_p(n^{-1/2})$ plus a remainder of smaller order.

The first term in the factor between square brackets in (B.1) is

$$n^{-1/2}X'u = n^{1/2}\sigma_{xu} + n^{1/2}(n^{-1}X'u - \sigma_{xu}), \tag{B.4}$$

so it can be decomposed in a deterministic $O(n^{1/2})$ and a random $O_p(1)$ component, see (A.2). To find the leading components in decreasing order of the second term in the factor between square brackets in (B.1) we start by considering $S_x^2 = \Sigma_x^2 + (S_x^2 - \Sigma_x^2)$, where $S_x^2 - \Sigma_x^2 = O_p(n^{-1/2})$, as shown in (A.4). Hence, using S_x and Σ_x as defined in Appendix A,

$$S_x = \Sigma_x + 0.5\Sigma_x^{-1}(S_x^2 - \Sigma_x^2) + o_p(n^{-1/2}), \tag{B.5}$$

which is proved by evaluating S_xS_x, as this yields $\Sigma_x^2 + S_x^2 - \Sigma_x^2 + o_p(n^{-1/2}) = S_x^2 + o_p(n^{-1/2})$.

Next we consider (2.10) evaluated in ρ_{xu}, which is

$$\hat{\sigma}_u^2(\rho_{xu}) = [1 - \rho_{xu}'S_x(n^{-1}X'X)^{-1}S_x\rho_{xu}]^{-1}[n^{-1}u'u - (n^{-1}X'u)'(n^{-1}X'X)^{-1}(n^{-1}X'u)]. \tag{B.6}$$

We first decompose $S_x(n^{-1}X'X)^{-1}S_x$ by substituting (B.3) and (B.5), which yields

$$\begin{aligned}
&S_x(n^{-1}X'X)^{-1}S_x \\
&= [\Sigma_x + 0.5\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)][\Sigma_{xx}^{-1} - \Sigma_{xx}^{-1}(n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1}] \\
&\qquad \times [\Sigma_x + 0.5\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)] + o_p(n^{-1/2}) \\
&= \Sigma_x\Sigma_{xx}^{-1}\Sigma_x + 0.5\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\Sigma_{xx}^{-1}\Sigma_x - \Sigma_x\Sigma_{xx}^{-1}(n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1}\Sigma_x \\
&\quad + 0.5\Sigma_x\Sigma_{xx}^{-1}\Sigma_x^{-1}(S_x^2 - \Sigma_x^2) + o_p(n^{-1/2}).
\end{aligned}$$

As diagonal matrices commute, we have $\Sigma_x^{-1}(S_x^2 - \Sigma_x^2) = (S_x^2 - \Sigma_x^2)\Sigma_x^{-1}$. Defining $\theta = 1 - \rho_{xu}'\Sigma_x\Sigma_{xx}^{-1}\Sigma_x\rho_{xu}$, we now obtain

$$\begin{aligned}
1 - \rho_{xu}'S_x(n^{-1}X'X)^{-1}S_x\rho_{xu} = \theta &- \rho_{xu}'\Sigma_x\Sigma_{xx}^{-1}\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\rho_{xu} \\
&+ \rho_{xu}'\Sigma_x\Sigma_{xx}^{-1}(n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1}\Sigma_x\rho_{xu} + o_p(n^{-1/2}).
\end{aligned}$$

Next the first factor of (B.6) can be decomposed as

$$\begin{aligned}
[1 - \rho_{xu}'S_x(n^{-1}X'X)^{-1}S_x\rho_{xu}]^{-1} = \theta^{-1} &+ \theta^{-2}\rho_{xu}'\Sigma_x\Sigma_{xx}^{-1}\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\rho_{xu} \\
&- \theta^{-2}\rho_{xu}'\Sigma_x\Sigma_{xx}^{-1}(n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1}\Sigma_x\rho_{xu} \\
&+ o_p(n^{-1/2}),
\end{aligned}$$

and for the second factor of (B.6) we find, using (B.4) and (B.3),

$$\begin{aligned}
&n^{-1}u'u - (n^{-1}X'u)'(n^{-1}X'X)^{-1}(n^{-1}X'u) \\
&= \sigma_u^2 + (n^{-1}u'u - \sigma_u^2) \\
&\qquad - [\sigma_{xu} + (n^{-1}X'u - \sigma_{xu})]'[\Sigma_{xx}^{-1} - \Sigma_{xx}^{-1}(n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1}] \\
&\qquad\quad \times [\sigma_{xu} + (n^{-1}X'u - \sigma_{xu})] + o_p(n^{-1/2}) \\
&= \sigma_u^2\theta + (n^{-1}u'u - \sigma_u^2) - 2\sigma_{xu}'\Sigma_{xx}^{-1}(n^{-1}X'u - \sigma_{xu}) \\
&\qquad + \sigma_{xu}'\Sigma_{xx}^{-1}(n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1}\sigma_{xu} + o_p(n^{-1/2}),
\end{aligned}$$

where we used $\sigma_{xu} = \sigma_u \Sigma_x \rho_{xu}$. The above yields for (B.6)

$$
\begin{aligned}
&\hat{\sigma}_u^2(\rho_{xu}) \\
&= \{\theta^{-1} + \theta^{-2}[\rho'_{xu}\Sigma_x\Sigma_{xx}^{-1}\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\rho_{xu} - \rho'_{xu}\Sigma_x\Sigma_{xx}^{-1}(n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1}\Sigma_x\rho_{xu}]\}\times \\
&\quad [\sigma_u^2\theta + (n^{-1}u'u - \sigma_u^2) - 2\sigma'_{xu}\Sigma_{xx}^{-1}(n^{-1}X'u - \sigma_{xu}) + \sigma'_{xu}\Sigma_{xx}^{-1}(n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1}\sigma_{xu}] \\
&\quad + o_p(n^{-1/2}) \\
&= \sigma_u^2 + \theta^{-1}(n^{-1}u'u - \sigma_u^2) - 2\theta^{-1}\sigma'_{xu}\Sigma_{xx}^{-1}(n^{-1}X'u - \sigma_{xu}) + \theta^{-1}\sigma'_{xu}\Sigma_{xx}^{-1}(n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1}\sigma_{xu} \\
&\quad + \sigma_u^2\theta^{-1}[\rho'_{xu}\Sigma_x\Sigma_{xx}^{-1}\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\rho_{xu} - \rho'_{xu}\Sigma_x\Sigma_{xx}^{-1}(n^{-1}X'X - \Sigma_{xx})\Sigma_{xx}^{-1}\Sigma_x\rho_{xu}] + o_p(n^{-1/2}) \\
&= \sigma_u^2 + \theta^{-1}(n^{-1}u'u - \sigma_u^2) - 2\theta^{-1}\sigma'_{xu}\Sigma_{xx}^{-1}(n^{-1}X'u - \sigma_{xu}) + \sigma_u^2\theta^{-1}\rho'_{xu}\Sigma_x\Sigma_{xx}^{-1}\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\rho_{xu} \\
&\quad + o_p(n^{-1/2}),
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
\hat{\sigma}_u(\rho_{xu}) = \sigma_u &+ 0.5\sigma_u^{-1}\theta^{-1}(n^{-1}u'u - \sigma_u^2) - \sigma_u^{-1}\theta^{-1}\sigma'_{xu}\Sigma_{xx}^{-1}(n^{-1}X'u - \sigma_{xu}) \\
&+ 0.5\sigma_u\theta^{-1}\rho'_{xu}\Sigma_x\Sigma_{xx}^{-1}\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\rho_{xu} + o_p(n^{-1/2}).
\end{aligned}
$$

For the factor between square brackets at the right-hand side of (B.1) we find now from the above, upon collecting all $o_p(1)$ terms in a remainder term,

$$
\begin{aligned}
&n^{-1/2}X'u - n^{1/2}\hat{\sigma}_u(\rho_{xu})S_x\rho_{xu} \\
&= n^{1/2}\sigma_{xu} + n^{1/2}(n^{-1}X'u - \sigma_{xu}) \\
&\quad - n^{1/2}[\sigma_u + 0.5\sigma_u^{-1}\theta^{-1}(n^{-1}u'u - \sigma_u^2) - \sigma_u^{-1}\theta^{-1}\sigma'_{xu}\Sigma_{xx}^{-1}(n^{-1}X'u - \sigma_{xu}) \\
&\qquad + 0.5\sigma_u\theta^{-1}\rho'_{xu}\Sigma_x\Sigma_{xx}^{-1}\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\rho_{xu}][\Sigma_x + 0.5\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)]\rho_{xu} + o_p(1) \\
&= n^{1/2}[\sigma_{xu} + (n^{-1}X'u - \sigma_{xu}) - \sigma_u\Sigma_x\rho_{xu} - 0.5\sigma_u^{-1}\theta^{-1}(n^{-1}u'u - \sigma_u^2)\Sigma_x\rho_{xu} \\
&\qquad + \sigma_u^{-1}\theta^{-1}\sigma'_{xu}\Sigma_{xx}^{-1}(n^{-1}X'u - \sigma_{xu})\Sigma_x\rho_{xu} \\
&\qquad - 0.5\sigma_u\theta^{-1}\rho'_{xu}\Sigma_x\Sigma_{xx}^{-1}\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\rho_{xu}\Sigma_x\rho_{xu} - 0.5\sigma_u\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\rho_{xu}] + o_p(1) \\
&= n^{1/2}[(n^{-1}X'u - \sigma_{xu}) + \sigma_u^{-1}\theta^{-1}\Sigma_x\rho_{xu}\sigma'_{xu}\Sigma_{xx}^{-1}(n^{-1}X'u - \sigma_{xu}) \\
&\qquad - 0.5\sigma_u\theta^{-1}\Sigma_x\rho_{xu}\rho'_{xu}\Sigma_x\Sigma_{xx}^{-1}\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\rho_{xu} - 0.5\sigma_u\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\rho_{xu} \\
&\qquad - 0.5\sigma_u^{-1}\theta^{-1}\Sigma_x\rho_{xu}(n^{-1}u'u - \sigma_u^2)] + o_p(1) \\
&= n^{1/2}[(I + \theta^{-1}\Sigma_x\rho_{xu}\rho'_{xu}\Sigma_x\Sigma_{xx}^{-1})(n^{-1}X'u - \sigma_{xu}) \\
&\qquad - 0.5\sigma_u(I + \theta^{-1}\Sigma_x\rho_{xu}\rho'_{xu}\Sigma_x\Sigma_{xx}^{-1})\Sigma_x^{-1}(S_x^2 - \Sigma_x^2)\rho_{xu} \\
&\qquad - 0.5\sigma_u^{-1}\theta^{-1}\Sigma_x\rho_{xu}(n^{-1}u'u - \sigma_u^2)] + o_p(1). \qquad \text{(B.7)}
\end{aligned}
$$

Note that the three explicit terms of (B.7) all have zero mean, are $O_p(1)$ and have a normal limiting distribution, according to our results of Appendix A. Hence, this expression has a limiting normal distribution too, say $[n^{-1/2}X'u - n^{1/2}\hat{\sigma}_u(\rho_{xu})S_x\rho_{xu}] \xrightarrow{d} \mathcal{N}(0, \sigma_u^2\Theta)$. Then, given (B.1), the limiting distribution of the inconsistency corrected OLS estimator is

$$
n^{1/2}[\hat{\beta}(\rho_{xu}) - \beta] \xrightarrow{d} \mathcal{N}(0, \sigma_u^2\Sigma_{xx}^{-1}\Theta\Sigma_{xx}^{-1}). \qquad \text{(B.8)}
$$

So, in order to establish this we should obtain Θ, the variance of the sum of all $O_p(1)$ terms of the vector (B.7).

Employing the asymptotic variances and covariances derived in Appendix A we find, when specializing for the case $\kappa_u = 3$ and $\kappa_x = 3$,

$$\begin{aligned}
\Theta = &(I + \theta^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x\Sigma_{xx}^{-1})(\Sigma_{xx} + \Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x)(I + \theta^{-1}\Sigma_{xx}^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x) \\
&+ 0.5(I + \theta^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x\Sigma_{xx}^{-1})\Sigma_x^{-1}\mathcal{R}(\Sigma_{xx} \circ \Sigma_{xx})\mathcal{R}\Sigma_x^{-1}(I + \theta^{-1}\Sigma_{xx}^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x) \\
&+ 0.5\theta^{-2}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x \\
&- (I + \theta^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x\Sigma_{xx}^{-1})\Sigma_{xx}\mathcal{R}^2(I + \theta^{-1}\Sigma_{xx}^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x) \\
&- (I + \theta^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x\Sigma_{xx}^{-1})\mathcal{R}^2\Sigma_{xx}(I + \theta^{-1}\Sigma_{xx}^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x) \\
&- \theta^{-1}(I + \theta^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x\Sigma_{xx}^{-1})\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x \\
&- \theta^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x(I + \theta^{-1}\Sigma_{xx}^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x) \\
&+ 0.5\theta^{-1}(I + \theta^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x\Sigma_{xx}^{-1})\mathcal{R}^2\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x \\
&+ 0.5\theta^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x\mathcal{R}^2(I + \theta^{-1}\Sigma_{xx}^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x).
\end{aligned}$$

First we shall evaluate this for the scalar case $K = 1$, where $\rho_{xu}^2 = 1 - \theta$. This yields $1 + \rho_{xu}^2/\theta = \theta^{-1}$, giving

$$\begin{aligned}
\sigma_x^{-2}\Theta &= (1 + \rho_{xu}^2/\theta)^2(1 + \rho_{xu}^2) + 0.5[(1 + \rho_{xu}^2/\theta)^2\rho_{xu}^2 + \rho_{xu}^2/\theta^2] \\
&\quad - 2[(1 + \rho_{xu}^2/\theta)^2\rho_{xu}^2 + \rho_{xu}^2(1 + \rho_{xu}^2/\theta)/\theta] + \rho_{xu}^4(1 + \rho_{xu}^2/\theta)/\theta \\
&= \theta^{-2}(1 + \rho_{xu}^2 + \rho_{xu}^2 - 2\rho_{xu}^2 - 2\rho_{xu}^2 + \rho_{xu}^4) \\
&= \theta^{-2}(1 - \rho_{xu}^2)^2 \\
&= 1,
\end{aligned}$$

which establishes the proof of Corollary 1.1.

For the case $K > 1$ such an elegant result proves to be an illusion, generally speaking. Denoting $\Phi = \Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x$ and using $\Phi\Sigma_{xx}^{-1}\Phi = (1 - \theta)\Phi$, we find for the first term of Θ the expression

$$\begin{aligned}
&(I + \theta^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x\Sigma_{xx}^{-1})(\Sigma_{xx} + \Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x)(I + \theta^{-1}\Sigma_{xx}^{-1}\Sigma_x\rho_{xu}\rho_{xu}'\Sigma_x) \\
&= (I + \theta^{-1}\Phi\Sigma_{xx}^{-1})(\Sigma_{xx} + \Phi)(I + \theta^{-1}\Sigma_{xx}^{-1}\Phi) \\
&= [\Sigma_{xx} + \Phi + \theta^{-1}\Phi + \theta^{-1}(1 - \theta)\Phi](I + \theta^{-1}\Sigma_{xx}^{-1}\Phi) \\
&= (\Sigma_{xx} + 2\theta^{-1}\Phi)(I + \theta^{-1}\Sigma_{xx}^{-1}\Phi) \\
&= \Sigma_{xx} + [2\theta^{-1} + \theta^{-1} + 2\theta^{-2}(1 - \theta)]\Phi \\
&= \Sigma_{xx} + (\theta^{-1} + 2\theta^{-2})\Phi,
\end{aligned}$$

and for the full expression

$$\begin{aligned}
\Theta &= \Sigma_{xx} + (\theta^{-1} + 2\theta^{-2})\Phi \\
&\quad + 0.5(I + \theta^{-1}\Phi\Sigma_{xx}^{-1})\Sigma_x^{-1}\mathcal{R}(\Sigma_{xx} \circ \Sigma_{xx})\mathcal{R}\Sigma_x^{-1}(I + \theta^{-1}\Sigma_{xx}^{-1}\Phi) + 0.5\theta^{-2}\Phi \\
&\quad - \Sigma_{xx}\mathcal{R}^2 - \theta^{-1}\Sigma_{xx}\mathcal{R}^2\Sigma_{xx}^{-1}\Phi - \theta^{-1}\Phi\mathcal{R}^2 - \theta^{-2}\Phi\mathcal{R}^2\Sigma_{xx}^{-1}\Phi \\
&\quad - \mathcal{R}^2\Sigma_{xx} - \theta^{-1}\mathcal{R}^2\Phi - \theta^{-1}\Phi\Sigma_{xx}^{-1}\mathcal{R}^2\Sigma_{xx} - \theta^{-2}\Phi\Sigma_{xx}^{-1}\mathcal{R}^2\Phi \\
&\quad - 2\theta^{-1}\Phi - 2\theta^{-2}\Phi\Sigma_{xx}^{-1}\Phi \\
&\quad + 0.5\theta^{-1}\mathcal{R}^2\Phi + 0.5\theta^{-2}\Phi\Sigma_{xx}^{-1}\mathcal{R}^2\Phi + 0.5\theta^{-1}\Phi\mathcal{R}^2 + 0.5\theta^{-2}\Phi\mathcal{R}^2\Sigma_{xx}^{-1}\Phi \\
&= \Sigma_{xx} + [\theta^{-1} + 2\theta^{-2} + 0.5\theta^{-2} - 2\theta^{-1} - 2\theta^{-2}(1-\theta)]\Phi + \\
&\quad + 0.5(I + \theta^{-1}\Phi\Sigma_{xx}^{-1})\mathcal{R}\Sigma_x^{-1}(\Sigma_{xx} \circ \Sigma_{xx})\Sigma_x^{-1}\mathcal{R}(I + \theta^{-1}\Sigma_{xx}^{-1}\Phi) \\
&\quad - \Sigma_{xx}\mathcal{R}^2 - \theta^{-1}\Sigma_{xx}\mathcal{R}^2\Sigma_{xx}^{-1}\Phi - 0.5\theta^{-1}\Phi\mathcal{R}^2 \\
&\quad - \mathcal{R}^2\Sigma_{xx} - 0.5\theta^{-1}\mathcal{R}^2\Phi - \theta^{-1}\Phi\Sigma_{xx}^{-1}\mathcal{R}^2\Sigma_{xx} - \theta^{-2}(\rho_{xu}'\Sigma_x\Sigma_{xx}^{-1}\Sigma_x\mathcal{R}^2\rho_{xu})\Phi \\
&= \Sigma_{xx} - \Sigma_{xx}\mathcal{R}^2 - \mathcal{R}^2\Sigma_{xx} - \theta^{-1}(\Sigma_{xx}\mathcal{R}^2\Sigma_{xx}^{-1}\Phi + \Phi\Sigma_{xx}^{-1}\mathcal{R}^2\Sigma_{xx}) \\
&\quad - 0.5\theta^{-1}(\Phi\mathcal{R}^2 + \mathcal{R}^2\Phi) + [\theta^{-1} + \theta^{-2}(0.5 - \rho_{xu}'\Sigma_x\Sigma_{xx}^{-1}\Sigma_x\mathcal{R}^2\rho_{xu})]\Phi \\
&\quad + 0.5(I + \theta^{-1}\Phi\Sigma_{xx}^{-1})\mathcal{R}\Sigma_x^{-1}(\Sigma_{xx} \circ \Sigma_{xx})\Sigma_x^{-1}\mathcal{R}(I + \theta^{-1}\Sigma_{xx}^{-1}\Phi).
\end{aligned}$$

C. Proof of Corollary 1.2

Evaluating Θ for the special case where $\rho_{xu} = (\rho_1, 0, ..., 0)'$ a simpler expression for the limiting variance is found, and again a very elegant solution for the coefficient of the endogenous regressor. Let e_1 be the $K \times 1$ vector with all its elements zero apart from the first one which is unity. Now, given this special case of ρ_{xu}, $\mathcal{R} = \rho_1 e_1 e_1'$ and $\Phi = \rho_1^2\sigma_1^2 e_1 e_1'$. Denoting $e_1'\Sigma_{xx}^{-1}e_1 = \sigma^{11}$ we have $\theta = 1 - \rho_1^2\sigma_1^2\sigma^{11}$. The we find $\Sigma_{xx}\mathcal{R}^2\Sigma_{xx}^{-1}\Phi = \Sigma_{xx}\rho_1^2 e_1 e_1'\Sigma_{xx}^{-1}\rho_1^2\sigma_1^2 e_1 e_1' = \rho_1^4\sigma_1^2\sigma^{11}\Sigma_{xx}e_1 e_1'$, $\Phi\mathcal{R}^2 = \rho_1^2\sigma_1^2 e_1 e_1'\rho_1^2 e_1 e_1' = \rho_1^4\sigma_1^2 e_1 e_1'$, $\rho_{xu}'\Sigma_x\Sigma_{xx}^{-1}\Sigma_x\mathcal{R}^2\rho_{xu} = \rho_1^4\sigma_1^2\sigma^{11}$ and $(I + \theta^{-1}\Phi\Sigma_{xx}^{-1})\mathcal{R} = (I + \theta^{-1}\rho_1^2\sigma_1^2 e_1 e_1'\Sigma_{xx}^{-1})\rho_1 e_1 e_1' = \theta^{-1}(\rho_1\theta + \rho_1^3\sigma_1^2\sigma^{11})e_1 e_1' = \theta^{-1}\rho_1 e_1 e_1'$. Substituting all these, we find

$$\begin{aligned}
\Theta &= \Sigma_{xx} - \rho_1^2(\Sigma_{xx}e_1e_1' + e_1e_1'\Sigma_{xx}) - \theta^{-1}\rho_1^4\sigma_1^2\sigma^{11}(\Sigma_{xx}e_1e_1' + e_1e_1'\Sigma_{xx}) - \theta^{-1}\rho_1^4\sigma_1^2 e_1e_1' \\
&\quad + \theta^{-1}(1 + 0.5\theta^{-1})\rho_1^2\sigma_1^2 e_1e_1' - \theta^{-2}\rho_1^6\sigma_1^4\sigma^{11}e_1e_1' + 0.5\theta^{-2}\rho_1^2\sigma_1^2 e_1e_1' \\
&= \Sigma_{xx} - \theta^{-1}(\rho_1^2\theta + \rho_1^4\sigma_1^2\sigma^{11})(\Sigma_{xx}e_1e_1' + e_1e_1'\Sigma_{xx}) \\
&\quad + \theta^{-2}(-\theta\rho_1^2 + \theta + 1 - \rho_1^4\sigma_1^2\sigma^{11})\rho_1^2\sigma_1^2 e_1e_1' \\
&= \Sigma_{xx} - \theta^{-1}\rho_1^2(\Sigma_{xx}e_1e_1' + e_1e_1'\Sigma_{xx}) + \theta^{-2}(\theta + 1 - \rho_1^2)\rho_1^2\sigma_1^2 e_1e_1'.
\end{aligned}$$

Hence, in this special case

$$\begin{aligned}
V(\rho_{xu}) &= \Sigma_{xx}^{-1}\Theta\Sigma_{xx}^{-1} \\
&= \Sigma_{xx}^{-1} - \theta^{-1}\rho_1^2(e_1e_1'\Sigma_{xx}^{-1} + \Sigma_{xx}^{-1}e_1e_1') + [\theta^{-1} + \theta^{-2}(1 - \rho_1^2)]\rho_1^2\sigma_1^2\Sigma_{xx}^{-1}e_1e_1'\Sigma_{xx}^{-1}.
\end{aligned} \tag{C.1}$$

We will now focus on the first element of estimator $\hat{\beta}(\rho_{xu})$. The OLS results $\hat{\beta}_1, \hat{u}$ and $\hat{\sigma}_u^2$ regarding β_1, u and σ_u^2 are all invariant under the model transformation given by $y = x_1\beta_1 + X_2\beta_2 + u = (M_2 + P_2)x_1\beta_1 + X_2\beta_2 + u = M_2x_1\beta_1 + X_2[(X_2'X_2)^{-1}X_2'\beta_1 + \beta_2] + u = x_1^*\beta_1 + X_2\beta_2^* + u$, where $P_2 = X_2(X_2'X_2)^{-1}X_2'$, $M_2 = I - P_2$ and $X_2'x_1^* = 0$.

Therefore, in the model with regressors (x_1^*, X_2), $\sigma_1^2 = n^{-1}x_1^{*\prime}x_1^*$ and $\sigma^{11} = \sigma_1^{-2}$, giving $\theta = 1 - \rho_1^2\sigma_1^2\sigma^{11} = 1 - \rho_1^2$. Then we find for $V(\rho_{xu})$ of (C.1) that

$$\begin{aligned} e_1'V(\rho_{xu})e_1 &= \sigma^{11} - 2\theta^{-1}\rho_1^2\sigma^{11} + [\theta^{-1} + \theta^{-2}(1-\rho_1^2)]\rho_1^2\sigma_1^2(\sigma^{11})^2 \\ &= \sigma^{11}[1 - 2\theta^{-1}\rho_1^2 + (\theta^{-1} + \theta^{-2} - \theta^{-2}\rho_1^2)(1-\theta)] \\ &= \sigma^{11}(1 - 2\theta^{-1}\rho_1^2 + \theta^{-1} + \theta^{-2} - \theta^{-2}\rho_1^2 - 1 - \theta^{-1} + \theta^{-1}\rho_1^2) \\ &= \theta^{-2}\sigma^{11}(1 - \theta\rho_1^2 - \rho_1^2) \\ &= \sigma^{11}, \end{aligned}$$

which is invariant with respect to ρ_{xu}.