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Artūras Juodis

www.ase.uva.nl/uva-econometrics

Amsterdam School of Economics
Department of Economics & Econometrics
Valckenierstraat 65-67
1018 XE AMSTERDAM
The Netherlands

UvA UNIVERSITEIT VAN AMSTERDAM

First Difference Transformation in Panel VAR models: Robustness, Estimation and Inference☆

Artūras Juodis[a,b]

[a] *Amsterdam School of Economics, University of Amsterdam, Valckenierstraat 65-67, 1018 XE, Amsterdam, The Netherlands*
[b] *Tinbergen Institute, The Netherlands*

Abstract

This paper considers estimation of Panel Vectors Autoregressive Models of order 1 (PVAR(1)) with possible cross-sectional heteroscedasticity in the error terms. We focus on fixed T consistent estimation methods in First Differences (FD) with additional strictly exogenous regressors. Additional results for the Panel FD OLS estimator and the FDLS type estimator of Han and Phillips (2010) are provided. In the covariance stationary case it is shown that the univariate moment conditions of the latter estimator are violated for general parameter matrices in the multivariate case. Furthermore, we simplify the analysis of Binder, Hsiao, and Pesaran (2005) by providing additional analytical results and extend the original model by taking into account possible cross-sectional heteroscedasticity and presence of strictly exogenous regressors. We show that in the three wave panel the log-likelihood function of the unrestricted TML estimator might violate the global identification assumption. The finite-sample performance of the analyzed methods is investigated in a Monte Carlo study.

Keywords: Dynamic Panel Data, Maximum Likelihood, Bias Correction, Fixed T Consistency, Monte Carlo Simulation.
JEL: C13, C33.

1. Introduction

When the feedback and interdependency between dependent variables and covariates is of particular interest, multivariate dynamic panel data models might arise as a natural modeling strategy. For example, particular policy measures can be seen as a response to the past evolution of the target quantity, meaning that the reduced form of two variables can be modeled by means of a Panel VAR (PVAR) model. In this paper we aim at providing a thorough analysis of the performance of fixed T consistent estimation techniques for PVARX(1) model based on observations in first differences. We

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Email address: `a.juodis@uva.nl` (Artūras Juodis)

will mainly focus on situations when the number of time periods is assumed to be relatively small, while the number of cross-section units is large.

The estimation of univariate dynamic panel data models and the incidental parameter problem of the ML estimators have received a lot of attention in the last three decades, see Nickell (1981), and Kiviet (1995) among others. However, a similar analysis for multivariate panel data models was not covered and investigated in detail. Main exceptions are papers by Holtz-Eakin et al. (1988), Hahn and Kuersteiner (2002), Binder, Hsiao, and Pesaran (2005, hereafter BHP) and Hayakawa (2013) presenting theoretical results for linear PVAR models.

Up to date most of the empirical papers in the field tend to use panel VAR methods for long cross-country panels, rather than short micro panels. For empirical examples of PVAR models for microeconomic panels, see Arellano (2003b, pp.116-120), Ericsson and Irandoust (2004), Koutsomanoli-Filippaki and Mamatzakis (2009) among others[1].

Because of the inconsistency of the Fixed effects (FE, ML) estimator, the estimation of dynamic panel data (DPD) models has been mainly concentrated within the GMM framework, with the version of the Arellano and Bond (1991) estimator and estimators of Arellano and Bover (1995), Blundell and Bond (1998), Ahn and Schmidt (1995) and Ahn and Schmidt (1997). However, Monte Carlo studies have revealed that the method of moments (MM) based estimators might be subject to substantial finite-sample biases, see Kiviet (1995), Alonso-Borrego and Arellano (1999) and BHP. Moreover, the finite sample properties are highly dependent on the particular choice of moment conditions imposed and/or weighting matrix used. These potentially unattractive finite sample properties of the GMM estimators have led to the recent interest in likelihood-based methods, that are corrected for incidental parameter bias. In this paper the ML estimator based on the likelihood function of the first differences of Hsiao et al. (2002), BHP and Kruiniger (2008) is analyzed

Monte Carlo results presented in BHP suggest that the Transformed likelihood based estimation developed in that paper outperforms the GMM based methods in terms of both finite sample bias and RMSE. However, the analysis performed there is limited in terms of both the methods analyzed and Monte Carlo designs considered. In particular, they did not consider cases where the models are stable, but the initial condition is not mean and/or covariance stationary. Furthermore, the Monte Carlo analysis was limited to situations where error terms are homoscedastic both in time and in the cross-section dimension, leaving the empirically relevant cases of heteroscedastic error terms unaddressed. We address both issues in the Monte Carlo designs presented in Section 5.

We aim to contribute to the literature in multiple ways. First of all, we show that the multivariate analogue of the FDOLS estimator of Han and Phillips (2010) is consistent only over a restricted parameter set. Secondly, we prove asymptotic normality of the TML estimator for models with cross-sectional heteroscedasticity and mean non-stationarity. Furthermore, we show that in the three wave panel the log-likelihood function of the unrestricted TML estimator can violate the global identification condition. Finally, the extensive Monte Carlo study expands the finite sample results available in the literature to cases with possible non-stationary initial conditions and cross-sectional heteroscedasticity.

The paper is structured as follows. In Section 2 we briefly present the model analyzed. Theoretical results for the FD and Transformed Maximum Likelihood estimators under different assumptions of the model are presented in Section 3. We continue in Section 4 with discussion about properties of

[1] Juessen and Linnemann (2010) provide a summary of empirical applications of PVAR with long panels.

the TML estimator when cross-sectional and/or time-series heteroscedasticity is present. In Section 5 we analyze finite sample performance of estimators considered in the paper by means of a Monte Carlo analysis. Finally, we conclude in Section 6.

Here we briefly discuss notation. **Bold** upper-case Greek letters are used to denote the original parameters, i.e. $\{\boldsymbol{\Phi}, \boldsymbol{\Sigma}, \boldsymbol{\Psi}\}$, while the lower-case Greek letters $\{\boldsymbol{\phi}, \boldsymbol{\sigma}, \boldsymbol{\psi}\}$ will denote $\operatorname{vec}(\cdot)$ ($\operatorname{vech}(\cdot)$ for symmetric matrices) of corresponding parameters, in the univariate setup corresponding parameters will be denoted by $\{\phi, \sigma^2, \psi^2\}$. We use $\rho(\boldsymbol{A})$ to denote the spectral radius[2] of a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. The commutation matrix $\mathbf{K}_{a,b}$ is defined such that for any $[a \times b]$ matrix $\boldsymbol{A}$, $\operatorname{vec}(\boldsymbol{A}') = \mathbf{K}_{a,b} \operatorname{vec}(\boldsymbol{A})$. The duplication matrix $\mathbf{D}_m$ is defined such that for symmetric $[a \times a]$ matrix $\operatorname{vec} \boldsymbol{A} = \mathbf{D}_m \operatorname{vech} \boldsymbol{A}$. We define $\bar{\boldsymbol{y}}_{i-} := (1/T)\sum_{t=1}^{T} \boldsymbol{y}_{i,t-1}$ and similarly $\bar{\boldsymbol{y}}_i := (1/T)\sum_{t=1}^{T} \boldsymbol{y}_{i,t}$. We will use $\tilde{x}$ to indicate variables after Within Group transformation (for example $\tilde{\boldsymbol{y}}_{i,t} = \boldsymbol{y}_{i,t} - \bar{\boldsymbol{y}}_i$), while $\ddot{\boldsymbol{x}}$ will be used for variables after a "quasi-averaging" transformation[3]. For further details regarding the notation used in this paper, see Abadir and Magnus (2002).

2. The Model

In this paper we consider the following PVAR(1) specification:

$$\boldsymbol{y}_{i,t} = \boldsymbol{\eta}_i + \boldsymbol{\Phi}\boldsymbol{y}_{i,t-1} + \boldsymbol{\varepsilon}_{i,t}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T, \tag{2.1}$$

where $\boldsymbol{y}_{i,t}$ is an $[m \times 1]$ vector, $\boldsymbol{\Phi}$ is an $[m \times m]$ matrix of parameters to be estimated, $\boldsymbol{\eta}_i$ is an $[m \times 1]$ vector of fixed effects and $\boldsymbol{\varepsilon}_{i,t}$ is an $[m \times 1]$ vector of innovations independent across i, with zero mean and constant covariance matrix $\boldsymbol{\Sigma}$.[4] If we set $m = 1$ the model reduces to the linear DPD model with AR(1) dynamics.

For the prototypical example of (2.1) consider the following bivariate model (see e.g. Arellano (2003a), Bun and Kiviet (2006) and Hayakawa and Pesaran (2012)):

$$\begin{aligned} y_{i,t} &= \eta_{yi} + \gamma y_{i,t-1} + \beta x_{i,t} + u_{i,t}, \\ x_{i,t} &= \eta_{xi} + \phi y_{i,t-1} + \rho x_{i,t-1} + v_{i,t}, \end{aligned}$$

where $\mathrm{E}[u_{i,t}v_{i,t}] = \sigma_{uv}$. This system has the following reduced form:

$$\begin{pmatrix} y_{i,t} \\ x_{i,t} \end{pmatrix} = \begin{pmatrix} \eta_{yi} + \beta\eta_{xi} \\ \eta_{xi} \end{pmatrix} + \begin{pmatrix} \gamma + \beta\phi & \beta\rho \\ \phi & \rho \end{pmatrix} \begin{pmatrix} y_{i,t-1} \\ x_{i,t-1} \end{pmatrix} + \begin{pmatrix} u_{i,t} + \beta v_{i,t} \\ v_{i,t} \end{pmatrix}. \tag{2.2}$$

Depending on the parameter values, the process $\{x_{i,t}\}_{t=0}^{T}$ can be either exogenous, weakly exogenous or endogenous. The process $\{x_{i,t}\}_{t=0}^{T}$ is strictly exogenous if $\phi = \sigma_{uv} = 0$, weakly exogenous if $\sigma_{uv} = 0$, and endogenous if $\sigma_{uv} \neq 0$.

For many empirically relevant applications the PVAR(1) model specification might be too restrictive and incomplete. In that case analysis is then extended to PVARX(1) model by including strictly exogenous variables in the model:

$$\boldsymbol{y}_{i,t} = \boldsymbol{\eta}_i + \boldsymbol{\Phi}\boldsymbol{y}_{i,t-1} + \boldsymbol{B}\boldsymbol{x}_{i,t} + \boldsymbol{\varepsilon}_{i,t}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T, \tag{2.3}$$

[2] $\rho(\boldsymbol{A}) := \max_i(|\lambda_i|)$, where λ_i's are (possibly complex) eigenvalues of a matrix $\boldsymbol{A}$.
[3] $\ddot{\boldsymbol{y}}_i = \bar{\boldsymbol{y}}_i - \boldsymbol{y}_{i,0}$ and $\ddot{\boldsymbol{y}}_{i-} = \bar{\boldsymbol{y}}_{i-} - \boldsymbol{y}_{i,0}$.
[4] Later in the paper we present the detailed analysis when $\boldsymbol{\Sigma}$ is not constant over cross-sectional units.

where $\boldsymbol{x}_{i,t}$ is a $[k \times 1]$ vector of strictly exogenous regressors and $\boldsymbol{B}$ is an $[m \times k]$ parameter matrix. Note that the model considered in Han and Phillips (2010) substantially differs from (2.3). They consider a model specification with lags of $\boldsymbol{x}_{i,t}$ and restricted parameters. Their specification can be accommodated within (2.3) only if the so-called *common factor* restrictions on $\boldsymbol{B}$ are imposed.

3. Theoretical results

At first we define several notions that will be primarily used for the model without exogenous regressors.

Definition 1 (Effect stationary initial condition)**.** The initial condition $\boldsymbol{y}_{i,0}$ is said to be effect stationary if:

$$\mathrm{E}[\boldsymbol{y}_{i,0}|\boldsymbol{\eta}_i] = (\boldsymbol{I}_m - \boldsymbol{\Phi}_0)^{-1}\boldsymbol{\eta}_i, \tag{3.1}$$

implying that the process $\{\boldsymbol{y}_{i,t}\}_{t=0}^{T}$ generated by (2.1) is effect stationary, $\mathrm{E}[\boldsymbol{y}_{i,t}|\boldsymbol{\eta}_i] = \mathrm{E}[\boldsymbol{y}_{i,0}|\boldsymbol{\eta}_i]$, for $\rho(\boldsymbol{\Phi}_0) < 1$.

Note that effect non-stationarity does not imply that the process $\{\boldsymbol{y}_{i,t}\}_{t=0}^{T}$ is mean non-stationary, i.e. $\mathrm{E}[\boldsymbol{y}_{i,t}] \neq \mathrm{E}[\boldsymbol{y}_{i,0}]$. The latter property of the process crucially depends on $\mathrm{E}[\boldsymbol{\eta}_i]$.

Definition 2 (Common dynamics)**.** The individual heterogeneity $\boldsymbol{\eta}_i$ is said to satisfy the *common dynamics assumption* if:

$$\boldsymbol{\eta}_i = (\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\boldsymbol{\mu}_i. \tag{3.2}$$

Under the common dynamics assumption, individual heterogeneity drops from the model in the pure unit root case $\boldsymbol{\Phi}_0 = \boldsymbol{I}_m$. Without this assumption the process $\{\boldsymbol{y}_{i,t}\}_{t=0}^{T}$ has a discontinuity at $\boldsymbol{I}_m$, as at this point the unrestricted process is a Multivariate Random Walk with drift. Combination of two notions results in $\mathrm{E}[\boldsymbol{y}_{i,0}|\boldsymbol{\mu}_i] = \boldsymbol{\mu}_i$, note that this term is well defined for $\rho(\boldsymbol{\Phi}_0) = 1$.

Definition 3 (Extensibility)**.** The DGP satisfies extensibility condition if:

$$\boldsymbol{\Phi}_0\boldsymbol{\Sigma}_0 = (\boldsymbol{\Phi}_0\boldsymbol{\Sigma}_0)'.$$

We call this condition"Extensibility" as in many situations it is a sufficient one to extend univariate conclusions to general $m \geq 1$ situations. One of the important implications of this condition is that:

$$\sum_{t=0}^{\infty}\boldsymbol{\Phi}_0^t\boldsymbol{\Sigma}_0(\boldsymbol{\Phi}_0^t)' = (\boldsymbol{I}_m - \boldsymbol{\Phi}_0^2)^{-1}\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}_0(\boldsymbol{I}_m - \boldsymbol{\Phi}_0^{2'})^{-1}.$$

Clearly, this condition is highly restrictive and uncommon in the literature, but as we will see from theoretical point of view this condition can be of a particular interest.

At first we will summarize the assumptions regarding the DGP used in this paper.

(A.1) The disturbances $\boldsymbol{\varepsilon}_{i,t}, t \leq T$, are i.i.d. for all i with finite fourth moment, with $\mathrm{E}[\boldsymbol{\varepsilon}_{i,t}] = \boldsymbol{0}_m$ and $\mathrm{E}[\boldsymbol{\varepsilon}_{i,t}\boldsymbol{\varepsilon}_{i,s}'] = 1_{(s=t)}\boldsymbol{\Sigma}_0$, $\boldsymbol{\Sigma}_0$ being a p.d. matrix.

(A.2) The initial deviation $\boldsymbol{u}_{i,0} := \boldsymbol{y}_{i,0} - \boldsymbol{\mu}_i$ is i.i.d. across cross-sectional units, with $\mathrm{E}[\boldsymbol{u}_{i,0}] = \boldsymbol{\gamma}_0$ with variance $\boldsymbol{\Psi}_{u,0}$, where $\boldsymbol{\eta}_i = (\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\boldsymbol{\mu}_i$ and a finite fourth moment.

(A.3) The following moment restrictions are satisfied: $\mathrm{E}[\boldsymbol{u}_{i,0}\boldsymbol{\varepsilon}'_{i,t}] = \mathbf{O}_m$ for all i and $t = \{1, \ldots, T\}$.

(A.4) $N \to \infty$, but T is fixed.

(A.5) Regressors (when present in the model) $\boldsymbol{x}_{i,t}$ are strictly exogenous $\mathrm{E}[\boldsymbol{\varepsilon}_{i,t}\boldsymbol{x}'_{i,s}] = \mathbf{O}_{m\times k}$, $\forall t, s = \{1, \ldots, T\}$ with a finite fourth moment.

(A.6) Matrix $\boldsymbol{\Phi}_0 \in \mathbb{R}^{m\times m}$ satisfies $\rho(\boldsymbol{\Phi}_0) < 1$.

(A.6)* Denote by $\boldsymbol{\kappa}$ a $[dim(\boldsymbol{\kappa}) \times 1]$ vector of unknown coefficients. $\boldsymbol{\kappa} \in \boldsymbol{\Gamma}$, where $\boldsymbol{\Gamma}$ is a compact subset of $\mathbb{R}^{dim(\boldsymbol{\kappa})}$ and $\boldsymbol{\kappa}_0 \in \text{interior}(\boldsymbol{\Gamma})$.

We will denote the set of Assumptions **(A.1)-(A.6)** by **SA** and by **SA*** set when in addition the **(A.6)*** assumption is satisfied. **SA** assumptions are used to establish results for the Panel FD estimators, while **SA*** are used to study asymptotic properties of the TML estimator. Assumption **(A.6)** is needed to ensure that the Hessian of the TML estimator has a full rank[5] in the model without regressors. On the other hand, in Assumption **(A.6)*** we implicitly extend the parameter space for $\boldsymbol{\Phi}$ to satisfy the usual compactness assumption so that both consistency and asymptotic normality can be proved directly, assuming the model is globally identified over the parameter space. However, as we will show in Section 3.3.3, the extended parameter space (beyond stationary region) might violate the global identification condition. The dimension of $\boldsymbol{\kappa}$ is left unspecified and will depend on a particular parametetrization used for estimation (with/without exogenous regressors, with/without mean term, etc. Note that in Assumption **(A.2)** no restrictions are imposed on $\boldsymbol{\mu}_i$ directly, but rather on the initial deviation $\boldsymbol{u}_{i,0}$, that in principle can be linear or non-linear function of $\boldsymbol{\mu}_i$.

In Section 4.1 we will consider the situation where we allow for individual specific $\boldsymbol{\Psi}_{u,0}$ and $\boldsymbol{\Sigma}_0$ matrices.

3.1. OLS in first differences

Original model in levels contains individuals effects that we remove from the model using the first-difference transformation. In that case the model specification is given by:

$$\Delta \boldsymbol{y}_{i,t} = \boldsymbol{\Phi}\Delta \boldsymbol{y}_{i,t-1} + \boldsymbol{B}\Delta \boldsymbol{x}_{i,t} + \Delta \boldsymbol{\varepsilon}_{i,t}, \quad t = 2, \ldots, T, i = 1, \ldots, N.$$

Before analyzing the FDOLS estimator for models with strictly exogenous regressors we define the following variables:

$$\Delta \boldsymbol{w}_{i,t} := \begin{pmatrix} \Delta \boldsymbol{y}_{i,t-1} \\ \Delta \boldsymbol{x}_{i,t} \end{pmatrix}, \quad \boldsymbol{S}_N := \left(\frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \Delta \boldsymbol{w}_{i,t} \Delta \boldsymbol{w}'_{i,t} \right),$$
$$\boldsymbol{\Sigma}_W := \operatorname*{plim}_{N\to\infty} \boldsymbol{S}_N, \quad \boldsymbol{\Upsilon} := (\boldsymbol{\Phi}, \boldsymbol{B}).$$

[5] See e.g. Bond et al. (2005) for the univariate proof that the Hessian matrix of the TML estimator is singular at the unit root.

Then if we pool observations for all t and i, we can define the pooled panel first difference estimator (FDOLS) as:

$$\hat{\boldsymbol{\Upsilon}}' = \boldsymbol{S}_N^{-1}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\Delta\boldsymbol{w}_{i,t}\Delta\boldsymbol{y}_{i,t}'\right) \tag{3.3}$$

Similarly to the conventional Fixed Effects (FE) transformation, the FD transformation introduces correlation between the explanatory variable $\Delta\boldsymbol{y}_{i,t-1}$ and the modified error terms $\Delta\boldsymbol{\varepsilon}_{i,t}$. As a result this estimator is not fixed T consistent[6], with the asymptotic bias derived in Proposition 3.1.

Proposition 3.1. *Let* $\{\boldsymbol{y}_{i,t}\}_{t=1}^{T}$ *be generated by* (2.3) *and Assumptions* ***SA*** *be satisfied. Then:*

$$\underset{N\to\infty}{\operatorname{plim}}(\hat{\boldsymbol{\Upsilon}} - \boldsymbol{\Upsilon}_0)' = -(T-1)\boldsymbol{\Sigma}_W^{-1}\begin{pmatrix}\boldsymbol{\Sigma}_0\\ \mathbf{O}_{k\times m}\end{pmatrix}. \tag{3.4}$$

It is easy to see that the FDOLS is numerically equal to the FE estimator with $T = 2$, thus the asymptotic bias is identical as well. Furthermore, as long as $T \geq 2$ the bias correction approaches as in Kiviet (1995) and Bun and Carree (2005) are readily available for this estimator.

Algorithm 1 Iterative Bias-correction procedure FDOLS

1. For $k = 1$ to $k^{\max}$:
2. Given $\boldsymbol{\Upsilon}^{(k-1)}$ compute $\boldsymbol{\Upsilon}^{(k)} = \hat{\boldsymbol{\Upsilon}} + (T-1)\hat{\boldsymbol{\Sigma}}(\boldsymbol{\Upsilon}^{(k-1)})\boldsymbol{S}_N^{-1}$;
3. If $\|\boldsymbol{\Upsilon}^{(k)} - \boldsymbol{\Upsilon}^{(k-1)}\| < \epsilon$, stop. For some pre-specified matrix norm $\|\cdot\|$.

To initialize iterations we set $\boldsymbol{\Upsilon}^{(0)} = \hat{\boldsymbol{\Upsilon}}$, and $\hat{\boldsymbol{\Sigma}}(\boldsymbol{\Upsilon}^{(k-1)})$ is defined as:

$$\hat{\boldsymbol{\Sigma}}(\boldsymbol{\Upsilon}) = \frac{1}{2N(T-1)}\sum_{i=1}^{N}\left(\sum_{t=2}^{T}\left(\Delta\boldsymbol{y}_{i,t} - \boldsymbol{\Upsilon}\Delta\boldsymbol{w}_{i,t}\right)\left(\Delta\boldsymbol{y}_{i,t} - \boldsymbol{\Upsilon}\Delta\boldsymbol{w}_{i,t}\right)'\right). \tag{3.5}$$

Asymptotic normality of the estimator can be proved by treating it as the solution of the following estimating equations:

$$\sum_{i=1}^{N}\sum_{t=2}^{T}\left((\Delta\boldsymbol{y}_{i,t} - \boldsymbol{\Upsilon}\Delta\boldsymbol{w}_{i,t})\Delta\boldsymbol{w}_{i,t}' + \frac{1}{2}(\Delta\boldsymbol{y}_{i,t} - \boldsymbol{\Upsilon}\Delta\boldsymbol{w}_{i,t})(\Delta\boldsymbol{y}_{i,t} - \boldsymbol{\Upsilon}\Delta\boldsymbol{w}_{i,t})'\boldsymbol{S}\right) = \mathbf{O}_{m\times(k+m)}, \tag{3.6}$$

where $\boldsymbol{S} = [\boldsymbol{I}_m \quad \mathbf{O}_{m\times k}]$.

Proposition 3.2. *Let Assumptions* ***SA*** *be satisfied and the iterative procedure in Algorithm 1 has the unique fixed point. Then:*

$$\sqrt{N}\left(\hat{\boldsymbol{\upsilon}}_{iBC} - \boldsymbol{\upsilon}_0\right) \xrightarrow{d} \mathrm{N}_m(\mathbf{0}_{m^2}, \mathfrak{F}), \tag{3.7}$$

[6] Actually not even large T consistent.

where:

$$\mathfrak{F} := \boldsymbol{V}^{-1}\mathfrak{X}\boldsymbol{V}^{-1}, \qquad \boldsymbol{V} = (\boldsymbol{\Sigma}_\Delta \otimes \boldsymbol{I}_m) - \frac{1}{2}(\boldsymbol{I}_{m(k+m)} + \mathbf{K}_{m,(k+m)})((\boldsymbol{S}'\boldsymbol{\Sigma}_0\boldsymbol{S}) \otimes \boldsymbol{I}_m),$$

$$\mathfrak{X} := \operatorname*{plim}_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \operatorname{vec}\mathfrak{D}_i \left(\operatorname{vec}\mathfrak{D}_i\right)',$$

$$\mathfrak{D}_i := \sum_{t=2}^{T}\left((\Delta\boldsymbol{y}_{i,t} - \boldsymbol{\Upsilon}_0\boldsymbol{w}_{i,t})\boldsymbol{w}'_{i,t} + \frac{1}{2}(\Delta\boldsymbol{y}_{i,t} - \boldsymbol{\Upsilon}_0\boldsymbol{w}_{i,t})(\Delta\boldsymbol{y}_{i,t} - \boldsymbol{\Upsilon}_0\boldsymbol{w}_{i,t})'\boldsymbol{S}\right)$$

Consistency and asymptotic normality of this estimator crucially depend on existence of the unique fixed point[7]. As a result, similarly to the estimator of Bun and Carree (2005) for some population parameter values, this estimator might fail to converge. These finite sample issues stimulate us to look for other analytical bias-correction procedures that have desirable finite sample properties irrespective of the DGP parameter values and initialization $\boldsymbol{y}_{i,0}$. Some special cases for model without exogenous regressors are discussed in the next section.

3.1.1. No exogenous regressors

One special case can be obtained if the initial condition $\boldsymbol{y}_{i,0}$ is covariance stationary, as in this case $\boldsymbol{\Sigma}_W$ is given by:

$$(T-1)\left(\boldsymbol{\Sigma}_0 + (\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\left(\sum_{t=0}^{\infty}\boldsymbol{\Phi}_0^t\boldsymbol{\Sigma}_0(\boldsymbol{\Phi}_0^t)'\right)(\boldsymbol{I}_m - \boldsymbol{\Phi}_0)'\right).$$

In the univariate case it is well known that covariance stationarity of $y_{i,0}$ is a sufficient condition to obtain an analytical bias-corrected estimator. However, it is no longer sufficient for $m > 1$ and general matrices $\boldsymbol{\Phi}_0$ and $\boldsymbol{\Sigma}_0$. One special case for analytical bias-corrected estimator is obtained for $(\boldsymbol{\Phi}_0, \boldsymbol{\Sigma}_0)$ that satisfy the "extensibility" condition, so that:

$$\boldsymbol{\Sigma}_W = 2(T-1)\boldsymbol{\Sigma}_0\left(\boldsymbol{I}_m + \boldsymbol{\Phi}'_0\right)^{-1}.$$

The resulting fixed T consistent estimator for $\boldsymbol{\Phi}$ is then given by:

$$\hat{\boldsymbol{\Phi}}_{FDLS} = 2\hat{\boldsymbol{\Phi}}_\Delta + \boldsymbol{I}_m.$$

It can be similarly shown that this estimator is fixed T consistent if $\boldsymbol{\Phi}_0 = \boldsymbol{I}_m$ and the common dynamics assumption is satisfied. For $m = 1$, this estimator was analyzed by Han and Phillips (2010), who named it the First Difference Least-Squares (FDLS) estimator, and proved its consistency and asymptotic normality under various assumptions. It should be noted that the same estimator (or the moment conditions it is based on) has been studied earlier in the DPD literature, see Bond et al. (2005), Ramalho (2005), Hayakawa (2007), Kruiniger (2007).

[7]Note that asymptotic distribution of the estimator depends upon the choice of $\hat{\boldsymbol{\Sigma}}(\boldsymbol{\Phi})$. Different asymptotic distribution is obtained if instead of using the $\boldsymbol{\Sigma}$ estimator in (3.5) we can opt for the standard infeasible ML estimator: $\hat{\boldsymbol{\Sigma}}(\boldsymbol{\Upsilon}) = \frac{1}{N(T-1)}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1} - \boldsymbol{B}\tilde{\boldsymbol{x}}_{i,t})(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1} - \boldsymbol{B}\tilde{\boldsymbol{x}}_{i,t})'\right)$.

Proposition 3.3 (Asymptotic Normality FDLS). *Let DGP for covariance stationary $\boldsymbol{y}_{i,t}$ satisfy extensibility condition together with conditions of Proposition 3.1. Then:*

$$\sqrt{N}\left(\hat{\boldsymbol{\phi}}_{FDLS}-\boldsymbol{\phi}_0\right)\xrightarrow{d}\mathrm{N}_m(\mathbf{0}_{m^2},\mathfrak{F}), \tag{3.8}$$

where:

$$\mathfrak{F}:=(\boldsymbol{\Sigma}_W^{-1}\otimes\boldsymbol{I}_m)\mathfrak{X}(\boldsymbol{\Sigma}_W^{-1}\otimes\boldsymbol{I}_m),\quad \mathfrak{X}:=\operatorname*{plim}_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\operatorname{vec}\mathfrak{D}_i\left(\operatorname{vec}\mathfrak{D}_i\right)',$$
$$\mathfrak{D}_i:=\left(\sum_{t=2}^{T}(2\Delta\boldsymbol{y}_{i,t}+(\boldsymbol{I}_m-\boldsymbol{\Phi}_0)\Delta\boldsymbol{y}_{i,t-1})\Delta\boldsymbol{y}'_{i,t-1}\right).$$

Proof of Proposition 3.3 follows directly as an application of the standard Lindeberg-Lévy CLT (see e.g. White (2000) for a general reference on asymptotic results).

Note that if the extensibility condition is violated the multivariate analogue of the FDLS estimator is *not* fixed T consistent. In that case the moment conditions similar to Han and Phillips (2010) can be considered. However, for general $\boldsymbol{\Phi}_0$ and $\boldsymbol{\Sigma}_0$ matrices these moment conditions are non-linear in $\boldsymbol{\Phi}$ and require numerical optimization making this approach undesirable, because the closed-form estimator is the main advantage of FDLS estimator as compared to the TML estimator that we describe in the next section.

3.2. Transformed MLE

3.2.1. General results

Independently Hsiao et al. (2002) and Kruiniger (2002)[8] suggested to build the likelihood for a transformation of the original data, such that after the transformation the likelihood function is free from incidental parameters. In particular, the likelihood function for the first differences was analyzed. BHP extended the univariate analysis of Hsiao et al. (2002)/Kruiniger (2002) to the multivariate case, allowing for possible cointegration between endogenous regressors.

In order to estimate (2.3) using the TML estimator of BHP we need to fully describe the density function $f(\Delta\boldsymbol{y}_i|\Delta\boldsymbol{X}_i)$. The only thing that needs to be specified and not imposed directly by (2.3) is $\mathrm{E}[\Delta\boldsymbol{y}_{i,1}|\Delta\boldsymbol{X}_i]$, where $\Delta\boldsymbol{X}_i$ is a $[Tk\times 1]$ vector of stacked exogenous variables. Conditional mean assumption is actually stronger than necessary for consistency and asymptotic normality of the TML estimator so we follow the approach of Hsiao et al. (2002) and consider the following linear projection for the first observation:

(TX.D) $\mathrm{Proj}[\Delta\boldsymbol{y}_{i,1}|\Delta\boldsymbol{X}_i]=\boldsymbol{B}\Delta\boldsymbol{x}_{i,1}+\boldsymbol{G}\Delta\boldsymbol{X}_i^{\dagger}.$

Here we define vector $\Delta\boldsymbol{X}_i^{\dagger}=(1,\Delta\boldsymbol{X}_i')'$, while the projection residual is denoted by $\boldsymbol{v}_{i,1}$. Before proceeding define by $\Delta\boldsymbol{E}$

$$\Delta\boldsymbol{E}_i:=(\boldsymbol{I}_{Tm}-\boldsymbol{L}_T\otimes\boldsymbol{\Phi})\Delta\boldsymbol{Y}_i-(\boldsymbol{I}_T\otimes\boldsymbol{B})\Delta\boldsymbol{X}_i-\operatorname{vec}(\boldsymbol{G}\Delta\boldsymbol{X}_i^{\dagger}\boldsymbol{e}_1'),$$

[8] Later appeared in Kruiniger (2008).

where $\Delta \boldsymbol{Y}_i = \operatorname{vec}(\Delta \boldsymbol{y}_{i,1}, \ldots, \Delta \boldsymbol{y}_{i,T})$. Then assuming (conditional) joint normality of the error terms and the initial observation, the log-likelihood function (up to a constant) is of the following form:

$$\ell(\boldsymbol{\kappa}) = -\frac{N}{2} \log |\boldsymbol{\Sigma}_{\Delta \boldsymbol{\tau}}| - \frac{N}{2} \operatorname{tr} \left(\left(\boldsymbol{\Sigma}_{\Delta \boldsymbol{\tau}}^{-1} \right) \frac{1}{N} \sum_{i=1}^{N} \Delta \boldsymbol{E}_i \Delta \boldsymbol{E}_i' \right), \tag{3.9}$$

where $\boldsymbol{\kappa} = (\boldsymbol{\phi}', \boldsymbol{\sigma}', \boldsymbol{\psi}')'$ and $\boldsymbol{\Psi} = \mathrm{E}[\boldsymbol{v}_{i,1} \boldsymbol{v}_{i,1}']$. The $\boldsymbol{\Sigma}_{\Delta \boldsymbol{\tau}}$ matrix has a block tridiagonal structure, with $-\boldsymbol{\Sigma}$ on first lower and upper off-diagonal blocks, and $2\boldsymbol{\Sigma}$ on all but first (1,1) diagonal blocks. The first (1,1) block is set to $\boldsymbol{\Psi}$, which takes into account the fact that the variance of $\boldsymbol{v}_{i,1}$ is treated as a free parameter.

The log-likelihood function in (3.9) depends on a fixed number of parameters, and satisfies the usual regularity conditions. Therefore under **SA*** the maximizer of this log-likelihood function is consistent with limiting normal distribution as N tends to infinity. In its general form, the asymptotic variance-covariance matrix of this estimator has a "sandwich" form. As it was discussed in BHP the "sandwich" form allows for *root*-N consistent inference, when the normality assumption is violated.

We will show below that conditioning on exogenous variables in first differences leads to concentrated log-likelihood functions in $\boldsymbol{\phi}$ only.

Theorem 3.1. *Let Assumptions **SA*** and **(TX.D)** be satisfied. Then the log-likelihood function of BHP for model* (2.3) *can be rewritten as:*

$$\begin{aligned}
\ell(\boldsymbol{\kappa}) = &-\frac{N}{2} \left((T-1) \log |\boldsymbol{\Sigma}| + \operatorname{tr} \left(\boldsymbol{\Sigma}^{-1} \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} (\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi} \tilde{\boldsymbol{y}}_{i,t-1} - \boldsymbol{B} \tilde{\boldsymbol{x}}_{i,t})(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi} \tilde{\boldsymbol{y}}_{i,t-1} - \boldsymbol{B} \tilde{\boldsymbol{x}}_{i,t})' \right) \right) \\
&- \frac{N}{2} \left(\log |\boldsymbol{\Theta}| + \operatorname{tr} \left(\boldsymbol{\Theta}^{-1} \frac{T}{N} \sum_{i=1}^{N} (\ddot{\boldsymbol{y}}_i - \boldsymbol{G} \Delta \boldsymbol{X}_i^{\dagger} - \boldsymbol{\Phi} \ddot{\boldsymbol{y}}_{i-} - \boldsymbol{B} \ddot{\boldsymbol{x}}_i)(\ddot{\boldsymbol{y}}_i - \boldsymbol{G} \Delta \boldsymbol{X}_i^{\dagger} - \boldsymbol{\Phi} \ddot{\boldsymbol{y}}_{i-} - \boldsymbol{B} \ddot{\boldsymbol{x}}_i)' \right) \right),
\end{aligned}$$

where $\boldsymbol{\kappa} = (\boldsymbol{\phi}', \boldsymbol{\sigma}', \boldsymbol{\theta}', \operatorname{vec} \boldsymbol{B}', \operatorname{vec} \boldsymbol{G}')'$, $\boldsymbol{\Theta} := \boldsymbol{\Sigma} + T(\boldsymbol{\Psi} - \boldsymbol{\Sigma})$ *and* $\ddot{\boldsymbol{x}}_i := \bar{\boldsymbol{x}}_i - \boldsymbol{x}_{i,0}$.

Proof. In Appendix A.1.2. □

The main conclusion of Theorem 3.1 is that in the case where $\boldsymbol{\Psi}$ is unrestricted, both the score and the Hessian matrix of the log-likelihood function have closed form expressions. This fact is advantageous in terms of both analytical tractability and numerical optimization. That implies that there is *no need* to use involved algorithms of BHP in order to compute the inverse and the determinant of the block tridiagonal matrix $\boldsymbol{\Sigma}_{\Delta \boldsymbol{\tau}}$. In order to simplify the notation and save some space, we introduce a new variable:

$$\boldsymbol{\xi}_i(\boldsymbol{\kappa}) := \ddot{\boldsymbol{y}}_i - \boldsymbol{G} \Delta \boldsymbol{X}_i^{\dagger} - \boldsymbol{\Phi} \ddot{\boldsymbol{y}}_{i-} - \boldsymbol{B} \ddot{\boldsymbol{x}}_i.$$

Remark 3.1. The log-likelihood function in Theorem 3.1 can be expressed in terms of the log-likelihood function for observations in levels $\ell_l^c(\tilde{\boldsymbol{\kappa}})$ (Within group part), as:

$$\ell(\boldsymbol{\kappa}) = \ell_l^c(\tilde{\boldsymbol{\kappa}}) - \frac{N}{2} \left(\log |\boldsymbol{\Theta}| + \operatorname{tr} \left(\boldsymbol{\Theta}^{-1} \frac{T}{N} \sum_{i=1}^{N} \boldsymbol{\xi}_i(\boldsymbol{\kappa}) \boldsymbol{\xi}_i(\boldsymbol{\kappa})' \right) \right),$$

where the additional (Between group) term corrects for the fixed T inconsistency of the standard ML (FE) estimator. It is just a generalization of Kruiniger (2008) and Han and Phillips (2013) conclusions to PVARX(1) with respect to the functional form of $\ell(\boldsymbol{\kappa})$.

Using the definition of $\boldsymbol{\xi}_i(\boldsymbol{\kappa})$ variable[9], we can formulate the following result.

Proposition 3.4. *Let Assumptions* $\boldsymbol{SA*}$ *be satisfied. Then the score vector associated with the log-likelihood function of Theorem 3.1 is given by*[10]*:*

$$\nabla(\boldsymbol{\kappa}) = \begin{pmatrix} \operatorname{vec}\left(\boldsymbol{\Sigma}^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1} - \boldsymbol{B}\tilde{\boldsymbol{x}}_{i,t})\tilde{\boldsymbol{y}}'_{i,t-1} + T\boldsymbol{\Theta}^{-1}\sum_{i=1}^{N}\boldsymbol{\xi}_i(\boldsymbol{\kappa})\ddot{\boldsymbol{y}}'_{i-}\right) \\ \mathbf{D}'_m \operatorname{vec}\left(\frac{N}{2}(\boldsymbol{\Sigma}^{-1}(\boldsymbol{Z}_N(\boldsymbol{\kappa}) - (T-1)\boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1})\right) \\ \mathbf{D}'_m \operatorname{vec}\left(\frac{N}{2}(\boldsymbol{\Theta}^{-1}(\boldsymbol{M}_N(\boldsymbol{\kappa}) - \boldsymbol{\Theta})\boldsymbol{\Theta}^{-1})\right) \\ \operatorname{vec}\left(\boldsymbol{\Sigma}^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1} - \boldsymbol{B}\tilde{\boldsymbol{x}}_{i,t})\tilde{\boldsymbol{x}}'_{i,t} + T\boldsymbol{\Theta}^{-1}\sum_{i=1}^{N}\boldsymbol{\xi}_i(\boldsymbol{\kappa})\ddot{\boldsymbol{x}}'_{i}\right) \\ \operatorname{vec}\left(T\boldsymbol{\Theta}^{-1}\sum_{i=1}^{N}\boldsymbol{\xi}_i(\boldsymbol{\kappa})\Delta\boldsymbol{X}_i^{\dagger'}\right) \end{pmatrix}. \tag{3.10}$$

Furthermore, the score vector satisfies the usual regularity condition:

$$\mathrm{E}[\nabla(\boldsymbol{\kappa_0})] = \mathbf{0}.$$

Proof. In Appendix A.1.3. □

The dimension of the $\boldsymbol{\kappa}$ vector is substantial especially for moderate values of m and k, hence from numerical point of view, maximization with respect to all parameters might not be appealing. In what follows we shall show that it is possible to construct the concentrated log-likelihood function with respect to the $\boldsymbol{\phi}$ parameter only[11]. To simplify further notation we define the following variables (assuming $N > Tk$):

$$\dot{\boldsymbol{y}}_i := \ddot{\boldsymbol{y}}_i - \left(\sum_{i=1}^{N}\ddot{\boldsymbol{y}}_i\Delta\boldsymbol{X}_i^{\dagger'}\right)\left(\sum_{i=1}^{N}\Delta\boldsymbol{X}_i^{\dagger}\Delta\boldsymbol{X}_i^{\dagger'}\right)^{-1}\Delta\boldsymbol{X}_i^{\dagger},$$

$$\dot{\boldsymbol{y}}_{i-} := \ddot{\boldsymbol{y}}_{i-} - \left(\sum_{i=1}^{N}\ddot{\boldsymbol{y}}_{i-}\Delta\boldsymbol{X}_i^{\dagger'}\right)\left(\sum_{i=1}^{N}\Delta\boldsymbol{X}_i^{\dagger}\Delta\boldsymbol{X}_i^{\dagger'}\right)^{-1}\Delta\boldsymbol{X}_i^{\dagger},$$

$$\acute{\boldsymbol{y}}_{i,t} := \tilde{\boldsymbol{y}}_{i,t} - \left(\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{\boldsymbol{y}}_{i,t}\tilde{\boldsymbol{x}}'_{i,t}\right)\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{\boldsymbol{x}}_{i,t}\tilde{\boldsymbol{x}}'_{i,t}\right)^{-1}\tilde{\boldsymbol{x}}_{i,t},$$

$$\acute{\boldsymbol{y}}_{i,t-1} := \tilde{\boldsymbol{y}}_{i,t-1} - \left(\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{\boldsymbol{y}}_{i,t-1}\tilde{\boldsymbol{x}}'_{i,t}\right)\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{\boldsymbol{x}}_{i,t}\tilde{\boldsymbol{x}}'_{i,t}\right)^{-1}\tilde{\boldsymbol{x}}_{i,t}.$$

Using the newly defined variables the concentrated log-likelihood function for $\boldsymbol{\kappa}^c = \{\boldsymbol{\phi}', \boldsymbol{\sigma}', \boldsymbol{\theta}'\}'$ is given by:

$$\ell^c(\boldsymbol{\kappa}^c) = c - \frac{N}{2}\left((T-1)\log|\boldsymbol{\Sigma}| + \operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}(\acute{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\acute{\boldsymbol{y}}_{i,t-1})(\acute{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\acute{\boldsymbol{y}}_{i,t-1})'\right)\right)$$
$$- \frac{N}{2}\left(\log|\boldsymbol{\Theta}| + \operatorname{tr}\left(\boldsymbol{\Theta}^{-1}\frac{T}{N}\sum_{i=1}^{N}(\dot{\boldsymbol{y}}_i - \boldsymbol{\Phi}\dot{\boldsymbol{y}}_{i-})(\dot{\boldsymbol{y}}_i - \boldsymbol{\Phi}\dot{\boldsymbol{y}}_{i-})'\right)\right).$$

[9] Some other variables used in this section are defined in Appendix A.1, so we will not repeat it here.

[10] See also derivations in Mutl (2009).

[11] The key observation for this result is that, although $\boldsymbol{B}$ parameter enters both $\operatorname{tr}(\cdot)$ components, $\ddot{\boldsymbol{x}}_i$ belongs to the column space spanned by $\Delta\boldsymbol{X}_i^{\dagger}$. Hence after concentrating out $\boldsymbol{G}$, $\boldsymbol{B}$ is no longer present in the second term.

Continuing we can concentrate out both $\boldsymbol{\Sigma}$ and $\boldsymbol{\Theta}$ to obtain the concentrated log-likelihood function for the $\boldsymbol{\phi}$ parameter vector only.

$$\begin{aligned}\ell^c_\Delta(\boldsymbol{\phi}) = c &- \frac{N(T-1)}{2}\log\left|\frac{1}{N(T-1)}\sum_{i=1}^N\sum_{t=1}^T(\acute{\boldsymbol{y}}_{i,t}-\boldsymbol{\Phi}\acute{\boldsymbol{y}}_{i,t-1})(\acute{\boldsymbol{y}}_{i,t}-\boldsymbol{\Phi}\acute{\boldsymbol{y}}_{i,t-1})'\right| \\ &- \frac{N}{2}\log\left|\frac{T}{N}\sum_{i=1}^N(\dot{\boldsymbol{y}}_i-\boldsymbol{\Phi}\dot{\boldsymbol{y}}_{i-})(\dot{\boldsymbol{y}}_i-\boldsymbol{\Phi}\dot{\boldsymbol{y}}_{i-})'\right|.\end{aligned}$$

However, as there is no closed-form solution for $\hat{\boldsymbol{\Phi}}$, numerical routines should be used to maximize this concentrated likelihood function[12]. The corresponding FOC can be easily derived from Proposition 3.4 for the unrestricted model.

Remark 3.2. In the Online Appendix Juodis (2014) we derive the exact expression for the empirical Hessian matrix $\boldsymbol{\mathcal{H}}^N(\hat{\boldsymbol{\kappa}}_{TMLE})$ and show that this matrix as well as its inverse are not block-diagonal, hence the TMLE of $\hat{\boldsymbol{\Phi}}$ and $\hat{\boldsymbol{\Sigma}}$ (as well as $\hat{\boldsymbol{\Theta}}$) are not asymptotically independent[13]. Non block-diagonality of the covariance matrix is very important and needs to be taken into account while performing Impulse Response analysis, see Cao and Sun (2011) for further details.

3.3. PVAR(1)/AR(1) specific results

3.3.1. Likelihood function with imposed covariance-stationarity

If we assume that $\boldsymbol{u}_{i,0}$ come from the stationary distribution then the log-likelihood function is a function of two parameters $\boldsymbol{\kappa}^{cov} = \{\boldsymbol{\phi}, \boldsymbol{\sigma}\}$ only with $\boldsymbol{\Theta}$ being of the following form:

$$\boldsymbol{\Theta} = \boldsymbol{\Sigma} + T(\boldsymbol{I}_m - \boldsymbol{\Phi})\left(\sum_{t=0}^\infty \boldsymbol{\Phi}^t\boldsymbol{\Sigma}(\boldsymbol{\Phi}^t)'\right)(\boldsymbol{I}_m - \boldsymbol{\Phi})'.$$

Kruiniger (2008) presents asymptotic results for the univariate version of this estimator under a range of assumptions regarding types of convergence. Results for PVAR(1) can be proved similarly.

Proposition 3.5. *Let Assumptions* ***SA**** *be satisfied. Then the score vector associated with the log-likelihood function of Theorem 3.1 under covariance stationarity is given by*[14]*:*

$$\nabla(\boldsymbol{\kappa}^{cov}) = \begin{pmatrix} \operatorname{vec}(\boldsymbol{W}_N(\boldsymbol{\kappa}^{cov})) + \boldsymbol{S}_1'\operatorname{vec}(\frac{N}{2}(\boldsymbol{\Theta}^{-1}(\boldsymbol{M}_N(\boldsymbol{\kappa}^{cov})-\boldsymbol{\Theta})\boldsymbol{\Theta}^{-1})) \\ \mathbf{D}_m'\left(\operatorname{vec}(\frac{N}{2}(\boldsymbol{\Sigma}^{-1}(\boldsymbol{Z}_N(\boldsymbol{\kappa}^{cov})-(T-1)\boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1})) + \boldsymbol{S}_2'\operatorname{vec}(\frac{N}{2}(\boldsymbol{\Theta}^{-1}(\boldsymbol{M}_N(\boldsymbol{\kappa}^{cov})-\boldsymbol{\Theta})\boldsymbol{\Theta}^{-1}))\right)\end{pmatrix}. \tag{3.11}$$

[12]For PVAR(1) model with spatial dependence of autoregressive type as in Mutl (2009), both $\boldsymbol{\Theta}$ and $\boldsymbol{\Sigma}$ parameters can be concentrated out but not the spatial dependence parameter λ.

[13]This result is in sharp contrast to the pure time series VAR's where it can be easily shown that estimates are indeed asymptotically independent.

[14]Note that there is a mistake in the derivations of the $\boldsymbol{S}_1$ term in Mutl (2009).

Here we define $\boldsymbol{j} := \operatorname{vec}\boldsymbol{I}_m$ *and:*

$$\begin{aligned}
\boldsymbol{W}_N(\boldsymbol{\kappa}) &:= \boldsymbol{\Sigma}^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})\tilde{\boldsymbol{y}}'_{i,t-1} + T\boldsymbol{\Theta}^{-1}\sum_{i=1}^{N}(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}\ddot{\boldsymbol{y}}_{i-})\ddot{\boldsymbol{y}}'_{i-}\\
\boldsymbol{S}_1 &:= -T\left[\left(\boldsymbol{\sigma}'\,\mathrm{D}'_m[\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}'\otimes\boldsymbol{\Phi}']^{-1}\right)\otimes\boldsymbol{I}_{m^2}\right]\\
&\quad\times(\boldsymbol{I}_m\otimes\mathrm{K}_m\otimes\boldsymbol{I}_m)\left[\boldsymbol{I}_{m^2}\otimes(\boldsymbol{j}-\boldsymbol{\phi}) + (\boldsymbol{j}-\boldsymbol{\phi})\otimes\boldsymbol{I}_{m^2}\right]\\
&\quad+T\left[\left(\boldsymbol{\sigma}'\,\mathrm{D}'_m[\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}'\otimes\boldsymbol{\Phi}']^{-1}\right)\otimes\left[\left((\boldsymbol{I}_m-\boldsymbol{\Phi})\otimes(\boldsymbol{I}_m-\boldsymbol{\Phi})\right)[\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}\otimes\boldsymbol{\Phi}]^{-1}\right]\right]\\
&\quad\times(\boldsymbol{I}_m\otimes\mathrm{K}_m\otimes\boldsymbol{I}_m)\left[\boldsymbol{I}_{m^2}\otimes\boldsymbol{\phi} + \boldsymbol{\phi}\otimes\boldsymbol{I}_{m^2}\right],\\
\boldsymbol{S}_2 &:= \boldsymbol{I}_{m^2} + T\left[(\boldsymbol{I}_m-\boldsymbol{\Phi})\otimes(\boldsymbol{I}_m-\boldsymbol{\Phi})\right][\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}\otimes\boldsymbol{\Phi}]^{-1}.
\end{aligned}$$

Proof. In Appendix A.1.3. □

It can be easily seen that $\mathrm{E}[\nabla(\boldsymbol{\kappa}_0^{cov})] \neq \mathbf{0}_{m^2+(1/2)(m+1)m}$, unless the initial condition is indeed covariance stationary (that is in contrast with the conclusion of Proposition 3.4 for the unrestricted estimator). Thus violation of the covariance stationarity implies that the $\hat{\boldsymbol{\kappa}}^{cov}$ estimator is inconsistent. If covariance stationarity of $\boldsymbol{u}_{i,0}$ is imposed, it is no longer possible to construct the concentrated log-likelihood for $\boldsymbol{\phi}$ parameter only and a joint optimization over whole $\boldsymbol{\kappa}^{cov}$ parameter vector is required[15].

Remark 3.3. In their article, Han and Phillips (2013) illustrate possible problems of the TML estimator with imposed covariance stationarity near unity. They observe that the log-likelihood function can be ill-behaved and bimodal close to $\phi_0 = 1$. In this paper, we do not investigate this possibility of possible bimodality for PVAR model as the behaviour of the log-likelihood function close to unity is not of prime interest for us. Furthermore, the bimodality in Han and Phillips (2013) is not related with bimodality of the unrestricted TML estimator as it will be discussed in Section 3.3.3.

3.3.2. Misspecification of the mean parameter

Let us assume that a researcher does not acknowledge the fact that data in differences is mean non-stationary and just considers the log-likelihood function as in Subsection 3.2 without $\boldsymbol{\gamma}$ parameter in it. In this section we denote by $\dot{\boldsymbol{\kappa}} = (\dot{\boldsymbol{\phi}}', \dot{\boldsymbol{\sigma}}', \dot{\boldsymbol{\theta}}')'$

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\phi}_0, \quad \dot{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_0, \quad \dot{\boldsymbol{\theta}} = \boldsymbol{\sigma}_0 + T\operatorname{vech}\left[(\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\,\mathrm{E}[\boldsymbol{u}_{i,0}\boldsymbol{u}'_{i,0}](\boldsymbol{I}_m - \boldsymbol{\Phi}_0)'\right].$$

Using this notation we have the following result.

Proposition 3.6. *Let Assumptions* $\boldsymbol{SA^{**}}$ *be satisfied, except that* $\mathrm{E}[\boldsymbol{u}_{i,0}] = \boldsymbol{\gamma}_0 \neq \mathbf{0}_m$. *Then the estimator* $\hat{\boldsymbol{\kappa}}$ *obtained as a maximizer of* (3.9) *is consistent in a sense that* $\hat{\boldsymbol{\kappa}} \xrightarrow{p} \dot{\boldsymbol{\kappa}}$. *Furthermore, under these assumptions:*

$$\sqrt{N}\,(\hat{\boldsymbol{\kappa}} - \dot{\boldsymbol{\kappa}}) \xrightarrow{d} \mathrm{N}(\mathbf{0}, \mathfrak{B}_{ML}),$$

[15] Unless the parameter space for $\boldsymbol{\Phi}$ and $\boldsymbol{\Sigma}$ is such that the "extensibility condition" is satisfied, see univariate results in Han and Phillips (2013).

where:

$$\mathfrak{B}_{ML} = \boldsymbol{\mathcal{H}}_\ell^{-1}\boldsymbol{\mathcal{I}}_\ell\boldsymbol{\mathcal{H}}_\ell^{-1},$$

$$\boldsymbol{\mathcal{H}}_\ell = \lim_{N\to\infty} \mathrm{E}\left[-\frac{1}{N}\boldsymbol{\mathcal{H}}^N(\dot{\boldsymbol{\kappa}})\right], and \quad \boldsymbol{\mathcal{I}}_\ell = \lim_{N\to\infty} \mathrm{E}\left[\frac{1}{N}\sum_{i=1}^{N}\nabla^{(i)}(\dot{\boldsymbol{\kappa}})\nabla^{(i)}(\dot{\boldsymbol{\kappa}})'\right].$$

Proof. Observe that we would obtain the same $\operatorname{plim}_{N\to\infty}\frac{1}{N}\ell(\boldsymbol{\kappa})$ (without constant term) if $\boldsymbol{u}_{i,0}$ were i.i.d multivariate normal with $\mathrm{E}[\boldsymbol{u}_{i,0}] = \mathbf{0}_m$ and finite $\mathrm{E}[\boldsymbol{u}_{i,0}\boldsymbol{u}_{i,0}']$. Hence the identification and consequently "consistency" follows similarly as in Proposition 4.1. The standard Taylor expansion based proof for asymptotic normality applies as the expected value of score at $\dot{\boldsymbol{\kappa}}$ is zero. See Kruiniger (2002) for more detailed proof as well as Appendix A.2. □

Remark 3.4. Note that we can think of $\boldsymbol{\gamma}$ as a (restricted) time effect for $\Delta\boldsymbol{y}_{i,1}$. However, there is a major difference as compared to the model with general time effects (not caused by $\mathrm{E}[\boldsymbol{u}_{i,0}]$). Unlike the $\boldsymbol{\gamma}$ parameter the *non-inclusion* of unrestricted time effects results in inconsistency of the TML estimator, as in the latter case non-inclusion of time effects results in the misspecification of (2.3). On the other hand, the assumption on $\mathrm{E}[\boldsymbol{u}_{i,0}]$ is not a part of the model (2.3). As it was already discussed in BHP, inclusion of time effects is equivalent to cross-sectional demeaning of all $\Delta\boldsymbol{y}_{i,t}$ beforehand.

3.3.3. Identification and bimodality issues for three-wave panels

In this section we study the behavior of the log-likelihood function for the TML estimator with an unrestricted initial condition. Consistency and asymptotic normality of any ML estimator, among others, requires the assumption that the expected log-likelihood function has the *unique* maximum at the true value. As we shall prove in this section, this condition is possibly violated for the TML estimator with unrestricted initial condition for $T = 2$. For the ease of exposition we consider univariate setup as in Hsiao et al. (2002).

Recall that based on the definition of $\boldsymbol{\Theta}$ in Theorem 3.1, the true value of θ^2 is given by:

$$\theta_0^2 = \sigma_0^2 + T(1-\phi_0)^2\psi_{u,0}^2, \quad \psi_{u,0}^2 = \mathrm{E}[u_{i,0}^2].$$

We define two new variables that we will use in further discussions extensively:

$$\phi_p := 2\left(\frac{x-1}{x}\right) + \phi_0, \quad x := 1 + (1-\phi_0)^2\psi_{u,0}^2/\sigma_0^2 = \frac{1}{2}\left(\frac{\theta_0^2}{\sigma_0^2}+1\right),$$

where p stands for "pseudo". This notation will become obvious from the following theorem:

Theorem 3.2. *Let assumptions* ***SA**** *be satisfied. Then for all $\phi_0 \in (-1;1)$ and $T = 2$ the following holds:*

$$\operatorname*{plim}_{N\to\infty} \ell^c(\phi_0) = \operatorname*{plim}_{N\to\infty} \ell^c(\phi_p) \tag{3.12}$$

for any value of $\psi_{u,0}^2 > 0$. Consequently the expected log-likelihood function has two local maxima at:

$$\boldsymbol{\kappa}_0 = \left(\phi_0, \sigma_0^2, \theta_0^2\right)',$$
$$\boldsymbol{\kappa}_p = \left(\phi_p, \theta_0^2, \sigma_0^2\right)'.$$

Several remarks regarding the results in Theorem 3.2 are worth mentioning[16]. First of all, instead of proving the result using the concentrated log-likelihood function, it can be proved similarly by considering the expected log-likelihood function directly. Secondly, if the parameter space is expressed in terms of $\boldsymbol{\kappa} = (\phi, \sigma^2, \psi)'$, then the value of ψ in both sets is equal to $(\sigma_0^2 + \theta_0^2)/2$.

Remark 3.5. While deriving the result we assumed that $\mathrm{E}[u_{i,0}] = 0$ and γ is not included in the parameter set. If $\mathrm{E}[u_{i,0}] \neq 0$ then two cases are possible: a) misspecified log-likelihood function as in Section 3.3.2 is considered and the result remains unchanged b) γ parameter is included in the set of parameters and as a result Theorem 3.2 does not hold true. For intuition observe that in the latter case the trivial estimator $\hat{\phi} = (\sum_{i=1}^{N} \Delta y_{i,2})/(\sum_{i=1}^{N} \Delta y_{i,1})$ is consistent. However, the key observation for this special case is that the model does not contain time effects. If model does contain time effects, $\hat{\phi}$ is no longer consistent and consequentially the main result of this section is still valid after cross-sectional demeaning of the data.

Remark 3.6. In the covariance stationary case it can be shown that the conclusion of Theorem 3.2 extends to PVAR(1) if the extensibility condition is satisfied and in addition $\boldsymbol{\Phi}_0$ is symmetric. In particular, this condition is satisfied by *all* three stationary designs in BHP with the pseudo value equal to the identity matrix.

To get more intuition about the problem at hand we slightly rewrite the expression for ϕ_p, assuming without loss of generality that for some $\alpha \geq 0$ we can rewrite $\psi_{u,0}^2$ as:

$$\psi_{u,0}^2 = \alpha \frac{\sigma_0^2}{1 - \phi_0^2}.$$

Consecutively we can rewrite ϕ_p in the following way:

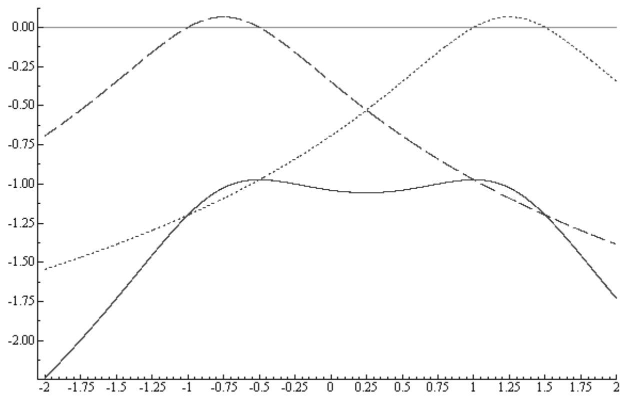
$$\phi_p = \frac{(\phi_0^2 + \phi_0)(1 - \alpha) + 2\alpha}{1 + \alpha + \phi_0(1 - \alpha)}. \tag{3.13}$$

From here it can be easily seen that then the pseudo-true value ϕ_p is equal to unity for covariance stationary initialization ($\alpha = 1$). Furthermore, we can consider other special cases:

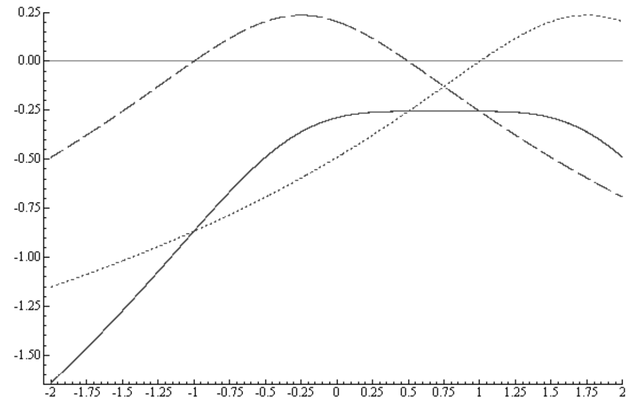
$$|\phi_0| \leq 1, \alpha = 0 \to \phi_p = \phi_0,$$
$$|\phi_0| \leq 1, \alpha \in (0, 1) \to \phi_0 < \phi_p < 1.$$

In Monte Carlo simulations researchers usually impose restrictions on the parameter space. In most cases ϕ is restricted to the stable region $(-1; 1)$, e.g. Hsiao et al. (2002). However, as it is clearly seen from Figure 2 (and derivations above) a stable region restriction on ϕ does not solve the bimodality issue and ϕ_p can lie in this interval.

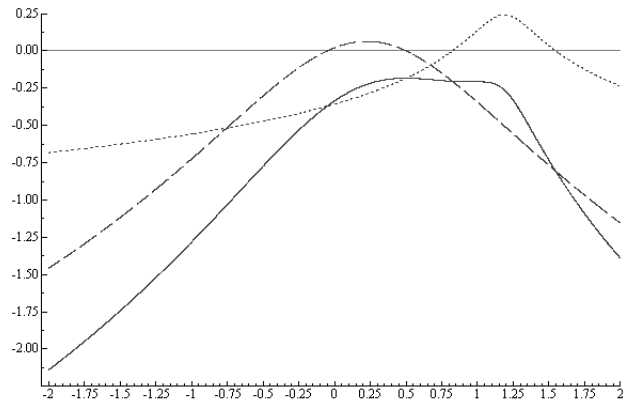
By construction the concentrated log-likelihood function is a sum of two *quasi*-concave functions with maxima at different points (Within Group and Between Group parts), bimodality does not disappear for $T > 2$. Thus by adding these two terms we end-up having function with possibly two modes, with the first one being of order $\mathcal{O}_p(NT)$ while the second one of order $\mathcal{O}_p(N)$. This different order of magnitude explains why for larger values of T the WG mode determines the shape of the whole function. To illustrate the problem described we present several figures of $\mathrm{plim}_{N\to\infty} \ell^c(\phi)$ for stationary initial conditions. The behavior of the concentrated log-likelihood function in Figures



(a) $\phi_0 = -0.5$, $T = 2$



(b) $\phi_0 = 0.5$, $T = 2$



(c) $\phi_0 = 0.5$, $T = 6$

Figure 1: Concentrated asymptotic log-likelihood function. In all figures the first mode is at the corresponding true value ϕ_0, while the second mode is located at $\phi = 1$. The initial observation is from covariance stationary distribution. The dashed line represents the WG part of the log-likelihood function, while the dotted line the BG part. The solid line, which stands for the log-likelihood function is a sum of dashed and dotted lines.

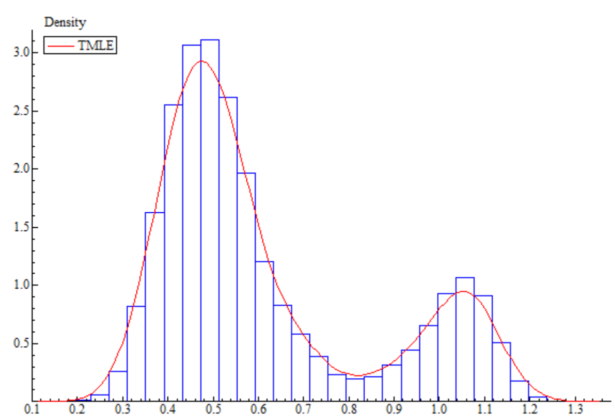


Figure 2: Histogram for the TMLE estimator with $T = 3$, $\boldsymbol{\phi}_0 = 0.5$, $N = 250$ and 10,000 MC replications. The initial observation is from covariance stationary distribution. Starting values for all iterations are set to $\phi^{(0)} = \{0.0, 0.1, \ldots, 1.5\}$. No non-negativity restrictions imposed.

1a, 1b and 1c is in line with the theoretical results provided earlier. Note that once ϕ_0 is approaching unity the log-likelihood function becomes flatter and flatter between the two points.

We can see from Figure 1c that once T is substantially bigger than 2, the "true value" mode starts to dominate the "pseudo value" mode. Based on all figures presented we can suspect that at least for covariance stationary initial conditions (or close to) the TML estimator will be biased *positively*, with the magnitude diminishing in T.

The main intuition behind the result in Theorem 3.2 is quite simple. When the log-likelihood function for θ (or ψ) is considered, no restrictions on the relative magnitude of those terms compared to σ^2 are imposed. In particular, it is possible that $\hat{\theta}^2 < \hat{\sigma}^2$ but that is a rather strange result given that:

$$\theta_0^2 = \sigma_0^2 + T(1-\phi_0)^2 \, \mathrm{E}[u_{i,0}^2].$$

But that is exactly what happens in the $\boldsymbol{\kappa}_p$ set as:

$$\theta_p^2 = \sigma_0^2, \quad \sigma_p^2 = \theta_0^2.$$

Hence the implicit estimate of $(1-\phi_0)^2 \, \mathrm{E}[u_{i,0}^2]$ is negative as we do *not* fully exploit the implied structure of $\operatorname{var} \Delta y_{i,1}$, which is a negative variance problem documented in panel data starting from the seminal paper of Maddala (1971)[17]. This problem was already encountered in some Monte Carlo studies performed in the literature (even for larger values of T), while some other authors only mention this possibility, e.g. Alvarez and Arellano (2003) and Arellano (2003a). For instance, Kruiniger (2008) mentions that for values of ϕ_0 close to unity the non-negative constraint on $(1-\phi_0)^2 \, \mathrm{E}[u_{i,0}^2]$, if imposed, is binding in 50 % of the cases. $\boldsymbol{\Theta}$ or $\boldsymbol{\Psi}$ parameter, on the other hand, is by construction p.d. (or non-negativity for univariate case). That explains why in some studies (for instance Ahn and Thomas (2006)) no numerical issues with the TML estimator were encountered. In this paper we analyze the limiting case of $T = 2$ and quantify the exact location of the second mode. Observations made in this section will provide intuition for some Monte Carlo results presented in Section 5.

4. Extensions

4.1. Cross-sectional Heterogeneity

In this subsection we consider model with possible cross-sectional heterogeneity in $\{\boldsymbol{\Sigma}, \boldsymbol{\Psi}_u\}$. For notational simplicity in this section we consider a model without exogenous regressors. All results presented can be extended to a model with exogenous regressors at the expense of more complicated notation.

(A.1)** The disturbances $\boldsymbol{\varepsilon}_{i,t}, t \leq T$, are i.h.d. for all i with $\mathrm{E}[\boldsymbol{\varepsilon}_{i,t}] = \mathbf{0}_m$ and $\mathrm{E}[\boldsymbol{\varepsilon}_{i,t}\boldsymbol{\varepsilon}_{i,s}] = 1_{s=t}\boldsymbol{\Sigma}_0(i)$, $\boldsymbol{\Sigma}_0(i)$ being p.d. matrix and $\max_i \mathrm{E}\left[\|\boldsymbol{\varepsilon}_{i,t}\|^{4+\delta}\right] < \infty$ for some $\delta > 0$.

(A.2)** The initial deviations $\boldsymbol{u}_{i,0}$ are i.h.d. across cross-sectional units, with $\mathrm{E}[\boldsymbol{u}_{i,0}] = \mathbf{0}_m$ and finite p.d. variance matrix $\boldsymbol{\Psi}_{u,0}(i)$ and $\max_i \mathrm{E}\left[\|\boldsymbol{u}_{i,0}\|^{4+\delta}\right] < \infty$, for some $\delta > 0$.

[16] We should emphasize that Theorem 3.2 has any theoretical meaning only if $\phi_p \in \boldsymbol{\Gamma}$.

[17] However, Maddala (1971) considers the Random Effects estimator for Dynamic Panel Data models, similarly to Alvarez and Arellano (2003).

We denote by $\check{\boldsymbol{\Sigma}}_0$ and similarly by $\check{\boldsymbol{\Psi}}_{u,0}$ the limiting values of corresponding sample averages, i.e. $\check{\boldsymbol{\Sigma}}_0 = \lim_{N\to\infty}(1/N)\sum_{i=1}^N \boldsymbol{\Sigma}_0(i)$[18]. Existence of the higher-order moments as presented in Assumptions **(A.1)**-(A.2)**** is a standard sufficient condition for the Lindeberg-Feller CLT to apply. We denote by **SA**** the set of assumptions **SA***, with **(A.1)-(A.2)** replaced by **(A.1)**-(A.2)****.

The univariate analogues of results presented in this section for the TMLE estimator, were derived by Kruiniger (2013) and recently extended by Hayakawa and Pesaran (2012).

The unrestricted log-likelihood function for $\boldsymbol{\kappa} = (\boldsymbol{\phi}', \boldsymbol{\sigma}(1)', \ldots, \boldsymbol{\sigma}(N)', \boldsymbol{\theta}(1)', \ldots, \boldsymbol{\theta}(N)')'$ suffers from the incidental parameter problem, as the number of parameters grows with the sample size, N. That implies that no N-consistent inference can be made on the $\boldsymbol{\sigma}(i)$ and $\boldsymbol{\theta}(i)$ parameters, but that does not imply that $\boldsymbol{\phi}$ parameter cannot be consistently estimated. Notably, we consider the *pseudo* log-likelihood function $\ell_p(\boldsymbol{\kappa})$[19]:

$$\begin{aligned}\ell_p(\boldsymbol{\kappa}) = &-\frac{N}{2}\left((T-1)\log|\boldsymbol{\Sigma}| + \operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\frac{1}{N}\sum_{i=1}^N\sum_{t=1}^T(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})'\right)\right)\\ &-\frac{N}{2}\left(\log|\boldsymbol{\Theta}| + \operatorname{tr}\left(\boldsymbol{\Theta}^{-1}\frac{T}{N}\sum_{i=1}^N(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}\ddot{\boldsymbol{y}}_{i-})(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}\ddot{\boldsymbol{y}}_{i-})'\right)\right),\end{aligned}$$

obtained if the researcher would mistakenly assume that observations are i.i.d. We shall prove that the conclusions from Section 3.2.1 continue to hold, with $\boldsymbol{\kappa}_0$ replaced by pseudo-true values $\check{\boldsymbol{\kappa}} = (\check{\boldsymbol{\phi}}', \check{\boldsymbol{\sigma}}', \check{\boldsymbol{\theta}}')'$, where:

$$\check{\boldsymbol{\sigma}} = \operatorname{vech}\check{\boldsymbol{\Sigma}}_0, \quad \check{\boldsymbol{\theta}} = \operatorname{vech}\check{\boldsymbol{\Theta}}_0, \quad \check{\boldsymbol{\phi}} = \boldsymbol{\phi}_0.$$

We assume that $\check{\boldsymbol{\kappa}}$ satisfy a compactness property similar to **(A.5)***. It is not difficult to see that the *point-wise* probability limit of $(1/N)\ell_p(\boldsymbol{\kappa})$ is given by:

$$\begin{aligned}\operatorname*{plim}_{N\to\infty}\frac{1}{N}\ell_p(\boldsymbol{\kappa}) = c &- \frac{1}{2}\left((T-1)\log|\boldsymbol{\Sigma}| + \operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\operatorname*{plim}_{N\to\infty}\boldsymbol{Z}_N(\boldsymbol{\kappa})\right)\right)\\ &- \frac{1}{2}\left(\log|\boldsymbol{\Theta}| + \operatorname{tr}\left(\boldsymbol{\Theta}^{-1}\operatorname*{plim}_{N\to\infty}\boldsymbol{M}_N(\boldsymbol{\kappa})\right)\right),\end{aligned}$$

where:

$$\begin{aligned}\operatorname*{plim}_{N\to\infty}\boldsymbol{Z}_N(\boldsymbol{\kappa}) &= (T-1)\check{\boldsymbol{\Sigma}}_0 + (\boldsymbol{\Phi}_0 - \boldsymbol{\Phi})\left(\operatorname*{plim}_{N\to\infty}\boldsymbol{R}_N\right)(\boldsymbol{\Phi}_0 - \boldsymbol{\Phi})' - \frac{1}{T}\left((\boldsymbol{\Phi}_0 - \boldsymbol{\Phi})\boldsymbol{\Xi}\check{\boldsymbol{\Sigma}}_0 + \check{\boldsymbol{\Sigma}}_0\boldsymbol{\Xi}'(\boldsymbol{\Phi}_0 - \boldsymbol{\Phi})'\right)\\ \operatorname*{plim}_{N\to\infty}\boldsymbol{M}_N(\boldsymbol{\kappa}) &= \check{\boldsymbol{\Theta}}_0 + (\boldsymbol{\Phi}_0 - \boldsymbol{\Phi})\left(\operatorname*{plim}_{N\to\infty}\boldsymbol{P}_N\right)(\boldsymbol{\Phi}_0 - \boldsymbol{\Phi})' + \frac{1}{T}\left((\boldsymbol{\Phi}_0 - \boldsymbol{\Phi})\boldsymbol{\Xi}\check{\boldsymbol{\Theta}}_0 + \check{\boldsymbol{\Theta}}_0\boldsymbol{\Xi}'(\boldsymbol{\Phi}_0 - \boldsymbol{\Phi})'\right).\end{aligned}$$

Note that we would obtain the same probability limit of the pseudo log-likelihood function if $\boldsymbol{u}_{i,0}$ and $\{\boldsymbol{\varepsilon}_{i,t}\}_{i=1,t=1}^{N,T}$ were i.i.d. Gaussian with parameters $\check{\boldsymbol{\kappa}}$, hence identification follows from the result

[18] As it was mentioned in Kruiniger (2013), Assumptions **(A.1)**-(A.2)**** are actually stronger than necessary, as it is sufficient to assume that $(1/N)\sum_{i=1}^N \mathrm{E}[\boldsymbol{\varepsilon}_{i,s}\boldsymbol{\varepsilon}_{i,s}'] = (1/N)\sum_{i=1}^N \mathrm{E}[\boldsymbol{\varepsilon}_{i,t}\boldsymbol{\varepsilon}_{i,t}']$ for all $s, t = 2, \ldots, T$ to prove consistency and asymptotic normality.

[19] Here "p" stands for pseudo and is used to distinguish from the standard TMLE log-likelihood function where inference on $\boldsymbol{\Sigma}$ and $\boldsymbol{\Theta}$ is possible.

for i.i.d. data. Similarly denote $\bar{\boldsymbol{\kappa}}_N = (\bar{\boldsymbol{\sigma}}'_N, \bar{\boldsymbol{\theta}}'_N, \bar{\boldsymbol{\phi}}')'$, where:

$$\bar{\boldsymbol{\sigma}}_N = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\sigma}_0(i), \quad \bar{\boldsymbol{\theta}}_N = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\theta}_0(i), \quad \bar{\boldsymbol{\phi}} = \boldsymbol{\phi}_0.$$

Using this result, consistency and asymptotic normality of $\hat{\boldsymbol{\kappa}}$ follows using standard arguments, see e.g. Amemiya (1986).

Proposition 4.1 (Consistency and Asymptotic normality)**.** *Under Assumptions **SA**** the maximizer of $\ell_p(\boldsymbol{\kappa})$ is consistent $\hat{\kappa} \xrightarrow{p} \check{\kappa}$. Furthermore, under these assumptions:*

$$\sqrt{N}\,(\hat{\boldsymbol{\kappa}} - \bar{\boldsymbol{\kappa}}_N) \xrightarrow{d} \mathrm{N}(\mathbf{0}, \mathfrak{B}_{PML}),$$

where:

$$\mathfrak{B}_{PML} = \boldsymbol{\mathcal{H}}_\ell^{-1}\boldsymbol{\mathcal{I}}_\ell\boldsymbol{\mathcal{H}}_\ell^{-1},$$
$$\boldsymbol{\mathcal{H}}_\ell = \lim_{N\to\infty}\mathrm{E}\left[-\frac{1}{N}\boldsymbol{\mathcal{H}}_p^N(\check{\boldsymbol{\kappa}})\right], \textit{and} \quad \boldsymbol{\mathcal{I}}_\ell = \lim_{N\to\infty}\frac{1}{N}\mathrm{E}\left[\sum_{i=1}^{N}\nabla_p^{(i)}(\boldsymbol{\kappa}_{0,i})\nabla_p^{(i)}(\boldsymbol{\kappa}_{0,i})'\right].$$

Here by $\nabla_p^{(i)}(\boldsymbol{\kappa}_{0,i})$ we denote the contribution of one cross-sectional unit i to the score of the pseudo log-likelihood function $\nabla_p(\bar{\boldsymbol{\kappa}})$ evaluated at the true values $\{\boldsymbol{\phi}_0, \boldsymbol{\sigma}_0(i), \boldsymbol{\theta}_0(i)\}$. Note that unless cross-sectional heterogeneity disappears as $N \to \infty$, the standard "sandwich" formula of the variance-covariance matrix evaluated at $\hat{\boldsymbol{\kappa}}$ is not a consistent estimate of the asymptotic variance-covariance matrix in Theorem 4.1, as in general:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\sigma}_0(i)\boldsymbol{\sigma}_0(i)' \neq \left(\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\sigma}_0(i)\right)\left(\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\sigma}_0(i)\right)', \tag{4.1}$$

while $\boldsymbol{\mathcal{H}}_\ell$ and $\mathfrak{B}_{PML}$ are not block-diagonal for fixed T. However, under some highly restrictive assumptions on higher order moments of initial observations and variance of strictly-exogenous regressors (when they are present) Hayakawa and Pesaran (2012) argue that it is possible to construct a modified consistent estimator of $\boldsymbol{\mathcal{I}}_\ell$ for the ARX(1) model. In the Monte Carlo section of this paper we will use the standard "sandwich" estimator for variance-covariance matrix without any modifications. We leave derivation of modified consistent estimator of $\boldsymbol{\mathcal{I}}_\ell$ for general PVARX(1) case for future research.

Remark 4.1. Note that by combining analysis in Propositions 3.6 and 4.1 we can see that for cases where $\mathrm{E}[\Delta\boldsymbol{y}_{i,1}] = \boldsymbol{\gamma}_i$ are individual specific, one can still obtain consistent estimate of $\boldsymbol{\Phi}$ erroneously assuming that the mean $\boldsymbol{\gamma}_i = \mathbf{0}_m$. On the other hand, the consistency of $\boldsymbol{\Phi}$ is not preserved if $\boldsymbol{\gamma}$ is included in the parameter set.

4.2. Time Series Heteroscedasticity

Unlike the case with cross-sectional heteroscedasticity, time-series heteroscedasticity results in inconsistent estimates of structural parameter matrix $\boldsymbol{\Phi}$. However, in this section we will show that for T sufficiently large one can still consistently estimate $\boldsymbol{\Phi}$[20]. At first we concentrate out the $\boldsymbol{\Theta}$ parameter and consider normalized version of the log-likelihood function:

$$\begin{aligned}\ell^c(\boldsymbol{\kappa}^c) = c - \frac{1}{2T}\log\left|\frac{T}{N}\sum_{i=1}^{N}(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}\ddot{\boldsymbol{y}}_{i-})(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}\ddot{\boldsymbol{y}}_{i-})'\right| \\ \frac{T-1}{2T}\log|\boldsymbol{\Sigma}| - \operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\frac{1}{2NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})'\right)\end{aligned}$$

As the term inside the first log-determinant term is of order $\mathcal{O}_p(T)$, the first component of the log-likelihood function is of order $o_p(1)$. Thus as $T \to \infty$ (for arbitrary N):

$$\ell^c(\boldsymbol{\kappa}^c) = c + o_p(1) + \frac{T-1}{2T}\log|\boldsymbol{\Sigma}| + \operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\frac{1}{2NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})'\right)$$

Clearly the remaining component is just the FE effect log-likelihood function and consistency of $\hat{\boldsymbol{\Sigma}}$ and $\hat{\boldsymbol{\Phi}}$ follows directly. For the case with time-series heteroscedasticity in $\boldsymbol{\Sigma}_t$ the log-likelihood function consistently estimates $\boldsymbol{\Sigma}_\infty := \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\Sigma}_t$ assuming that this limit exists.
The gradient of the log-likelihood function is given by:

$$\frac{1}{\sqrt{NT}}\nabla(\boldsymbol{\kappa}) = \frac{1}{\sqrt{NT}}\begin{pmatrix} \operatorname{vec}\left(\boldsymbol{\Sigma}^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})\tilde{\boldsymbol{y}}_{i,t-1}' + \left(\frac{1}{T}\hat{\boldsymbol{\Theta}}\right)^{-1}\sum_{i=1}^{N}(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}\ddot{\boldsymbol{y}}_{i-})\ddot{\boldsymbol{y}}_{i-}'\right) \\ \frac{1}{2}\mathbf{D}_m'\operatorname{vec}\left(\boldsymbol{\Sigma}^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\left((\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})' - \frac{T-1}{T}\boldsymbol{\Sigma}\right)\right)\boldsymbol{\Sigma}^{-1}\right)\end{pmatrix}.$$

As it was shown in the previous sections, the second component of the derivative w.r.t $\boldsymbol{\Phi}$ is of order $\mathcal{O}_p(\sqrt{N})$. As a result, under the assumption that $N/T \to \rho$ evaluated at the true value of $\boldsymbol{\Phi}_0$:

$$\begin{aligned}\frac{1}{\sqrt{NT}}\left(\frac{1}{T}\hat{\boldsymbol{\Theta}}\right)^{-1}\sum_{i=1}^{N}(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}_0\ddot{\boldsymbol{y}}_{i-})\ddot{\boldsymbol{y}}_{i-}' &= \sqrt{\rho}\left(\frac{1}{T}\hat{\boldsymbol{\Theta}}\right)^{-1}\frac{1}{N}\sum_{i=1}^{N}(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}_0\ddot{\boldsymbol{y}}_{i-})\ddot{\boldsymbol{y}}_{i-}' + o_p(1) \\ &= \sqrt{\rho}\left((\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\boldsymbol{\Psi}_{u,0}(\boldsymbol{I}_m - \boldsymbol{\Phi}_0)'\right)^{-1}\left[(\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\boldsymbol{\Psi}_{u,0}\right] + o_p(1), \\ &= \sqrt{\rho}\left(\boldsymbol{I}_m - \boldsymbol{\Phi}_0'\right)^{-1} + o_p(1).\end{aligned}$$

where the corresponding result is valid irrespective of whether time-series heteroscedasticity is present or not. Now consider the bias for the score of the fixed effects estimator evaluated at $\boldsymbol{\Phi}_0$ and

[20] In order to show similar results for more models with exogenous regressors one has to prove that as $T \to \infty$ the incidental parameter matrix $\boldsymbol{G}$ does not result in an incidental parameter problem.

$\bar{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\Sigma}_t$ (as in e.g. Juodis (2013)):

$$
\begin{aligned}
-\sqrt{\rho}T\bar{\boldsymbol{\Sigma}}^{-1}\,\mathrm{E}[\bar{\boldsymbol{\varepsilon}}_i\bar{\boldsymbol{y}}_i{}'] + o(1) &= -\frac{\sqrt{\rho}}{T}\bar{\boldsymbol{\Sigma}}^{-1}\left(\sum_{t=0}^{T-2}\left(\sum_{l=0}^{t}\boldsymbol{\Phi}_0^l\right)\boldsymbol{\Sigma}_{T-1-t}\right)' + o(1)\\
&= -\frac{\sqrt{\rho}}{T}\bar{\boldsymbol{\Sigma}}^{-1}\left((\boldsymbol{I}_m - \boldsymbol{\Phi}_0)^{-1}\sum_{t=0}^{T-2}\left(\boldsymbol{I}_m - \boldsymbol{\Phi}_0^{t+1}\right)\boldsymbol{\Sigma}_{T-1-t}\right)' + o(1)\\
&= -\sqrt{\rho}(\boldsymbol{I}_m - \boldsymbol{\Phi}_0')^{-1} + \frac{1}{T}\bar{\boldsymbol{\Sigma}}^{-1}\left(\sum_{t=0}^{T-2}\boldsymbol{\Phi}_0^{t+1}\boldsymbol{\Sigma}_{T-1-t}\right)' + o(1)\\
&= -\sqrt{\rho}(\boldsymbol{I}_m - \boldsymbol{\Phi}_0')^{-1} + o(1).
\end{aligned}
$$

Here the last line follows if one assumes that $\boldsymbol{\Sigma}_s$ process is bounded, so that the sum term is of order $\mathcal{O}(1)$. As a result the large N, T distribution of the TML estimator is identical to the one of the bias-corrected FE estimator of Hahn and Kuersteiner (2002).

In the previous section we have shown that in the correctly specified model with time-series homoscedasticity the score of the TML estimator *fully* removes the induced bias of the FE estimator. This conclusion was established based on the assumption that $N \to \infty$ for any *fixed* value of T. In this section we have extended this result by showing that under presence of possible time-series heteroscedasticity the estimating equations of the TML estimator remove the leading bias of the FE estimator.

5. Simulation Study

5.1. Monte Carlo Setup

At first we present the general DGP that can be used to generate initial conditions $\boldsymbol{y}_{i,0}$. We will distinguish between stability and stationarity conditions. We call the process $\{\boldsymbol{y}_{i,t}\}_{t=0}^{T}$ dynamically stable if $\rho(\boldsymbol{\Phi}) < 1$ and (covariance) stationary if in addition the first two moments are constant over time ($t = \{0, \ldots, T\}$).

$$
\boldsymbol{y}_{i,0} = \boldsymbol{a}_i + \boldsymbol{E}_i\boldsymbol{\mu}_i + \boldsymbol{C}_i\boldsymbol{\varepsilon}_{i,0}, \quad \boldsymbol{\varepsilon}_{i,0} \sim \mathrm{IID}\left(\boldsymbol{0}_m, \sum_{j=0}^{\infty}\boldsymbol{\Phi}_0^j\boldsymbol{\Sigma}_0(\boldsymbol{\Phi}_0^j)'\right), \tag{5.1}
$$

for some parameter matrices $\boldsymbol{a}_i$ $[m \times 1]$, $\boldsymbol{E}_i$ $[m \times m]$ and $\boldsymbol{C}_i$ $[m \times m]$. The special case of this setup is the (covariance) stationary model if $\boldsymbol{a}_i = \boldsymbol{0}_m$ and $\boldsymbol{C}_i = \boldsymbol{E}_i = \boldsymbol{I}_m$.

In what follows we will set $\boldsymbol{a}_i$ to $\boldsymbol{0}_2$ for all Designs considered[21]. As we only consider $\boldsymbol{\Phi}_0$ such that $\rho(\boldsymbol{\Phi}_0) < 1$, we (without loss of generality) generate the individual heterogeneity $\boldsymbol{\mu}_i$ (rather than $\boldsymbol{\eta}_i$) using similar procedure as in BHP:

$$
\boldsymbol{\mu}_i = \pi\left(\frac{q_i - 1}{\sqrt{2}}\right)\check{\boldsymbol{\eta}}_i, \quad q_i \overset{iid}{\sim} \chi^2(1), \quad \check{\boldsymbol{\eta}}_i \overset{iid}{\sim} \mathrm{N}(\boldsymbol{0}_2, \boldsymbol{\Sigma}_{\check{\boldsymbol{\eta}}}). \tag{5.2}
$$

[21] In the Online Appendix some additional results for Design 2 are presented with $\boldsymbol{a}_i = \boldsymbol{\imath}_2$.

Unlike in the paper of BHP we do not fix $\boldsymbol{\Sigma}_{\breve{\boldsymbol{\eta}}} = \boldsymbol{\Sigma}$, instead we extend the approach of Kiviet (2007) by specifying[22][23]:

$$\operatorname{vec} \boldsymbol{\Sigma}_{\breve{\boldsymbol{\eta}}} = \left(\frac{1}{T} \sum_{t=1}^{T} \left(\boldsymbol{\Phi}_0^t (\boldsymbol{E} - \boldsymbol{I}_m) + \boldsymbol{I}_m \right) \otimes \left(\boldsymbol{\Phi}_0^t (\boldsymbol{E} - \boldsymbol{I}_m) + \boldsymbol{I}_m \right) \right)^{-1} (\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}_0 \otimes \boldsymbol{\Phi}_0)^{-1} \operatorname{vec} \boldsymbol{\Sigma}_0. \quad (5.3)$$

The way we generate $\boldsymbol{\mu}_i$ ensures that the individual heterogeneity is *not normally* distributed, but still IID across individuals. In the effect stationary case the particular way the $\boldsymbol{\mu}_i$ are generated does not influence the behavior of TML log-likelihood function. However, the non-normality of $\boldsymbol{\mu}_i$ in the effect non-stationary case implies non-normality of $\boldsymbol{u}_{i,0}$ and, hence, a quasi maximum likelihood interpretation of the likelihood function. With respect to the error terms we restrict our attention to $\boldsymbol{\varepsilon}_{i,t}$ being normally distributed $\forall i, t$[24].

5.2. Designs

The parameter set which is common for all designs irrespective of their properties regarding stability and stationarity, consists of the triple $\{N; T; \pi\}$. We restrict the parameter space of $\{N; T; \pi\}$ to the following set:

$$N = \{100; 250\}, \quad T = \{3; 6\}, \quad \pi = \{1; 3\}.$$

In the DPD literature it is well known that in the effect stationary case a higher value π leads to diminishing finite sample properties of the GMM estimators, see e.g. Bun and Windmeijer (2010) and Bun and Kiviet (2006). That might have indirect influence on the TML estimator even in the effect stationary case, as we will use GMM estimators as starting values for numerical optimization of the log-likelihood function.

In this paper six different Monte Carlo designs are considered. One of them (1) is adapted from the original analysis of BHP, while the other five are supposed to reveal whether the proposed methods are robust with respect to different assumptions regarding the parameter matrix $\boldsymbol{\Phi}_0$, the initial conditions $\boldsymbol{y}_{i,0}$ and cross-sectional heteroscedasticity. In the case where observations are covariance stationary or cointegrated, BHP calibrated the design matrices $\boldsymbol{\Phi}$ and $\boldsymbol{\Sigma}$ such that the population $R^2_{\Delta l}$[25] remained approximately constant (≈ 0.237) between designs.

Design 1 (Covariance Stationary PVAR with $\rho(\boldsymbol{\Phi}_0) = 0.8$ from BHP)**.**

$$\boldsymbol{\Phi}_0 = \begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.6 \end{pmatrix}, \quad \boldsymbol{\Sigma}_0 = \begin{pmatrix} 0.07 & -0.02 \\ -0.02 & 0.07 \end{pmatrix}, \quad \boldsymbol{\Sigma}_{\breve{\boldsymbol{\eta}}} = \begin{pmatrix} 0.123 & 0.015 \\ 0.015 & 0.123 \end{pmatrix}.$$

[22] See Online Appendix.

[23] If variance of $\boldsymbol{\varepsilon}_{i,t}$ differs between individuals then we evaluate this expression at $\bar{\boldsymbol{\Sigma}}_n$ rather than at $\boldsymbol{\Sigma}$.

[24] The analysis can be easily extended to the cases where the error terms are skewed and/or have fatter tails as compared to the Gaussian distribution. As a partial robustness of their results BHP considered t- and chi square distributed disturbances, but the results were close to the Gaussian setup. The estimation output for these setups was not presented in their paper.

[25] Computation of the population R^2 for stationary series: $R^2_{\Delta l} = 1 - \frac{\boldsymbol{\Sigma}_{l,l}}{\boldsymbol{\Gamma}_{l,l}}, \quad l = 1$; where $\operatorname{vec}(\boldsymbol{\Gamma})$ in the covariance stationary case is given by: $\operatorname{vec}(\boldsymbol{\Gamma}) = \left(((\boldsymbol{I}_m - \boldsymbol{\Phi}_0) \otimes (\boldsymbol{I}_m - \boldsymbol{\Phi}_0)) (\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}_0 \otimes \boldsymbol{\Phi}_0)^{-1} + \boldsymbol{I}_{m^2} \right) \mathbf{D}_m \boldsymbol{\sigma}$.

The second eigenvalue is equal to 0.4 and the population R^2_Δ values are given by $R^2_{\Delta l} = 0.2396$, $l = 1, 2$.

Although the Monte Carlo designs in BHP are well chosen, they are quite limited in scopeas the analysis was mainly focused on the influence of $\rho(\boldsymbol{\Phi}_0)$. Furthermore, all design matrices in the stationary designs were assumed to be symmetric and Toeplitz, which substantially shrinks the parameter space of $\boldsymbol{\Phi}_0$ and $\boldsymbol{\Sigma}$.

Design 2 (Covariance Stationary PVAR with $\rho(\boldsymbol{\Phi}_0) = 0.50498$)**.**

$$\boldsymbol{\Phi}_0 = \begin{pmatrix} 0.4 & 0.15 \\ -0.1 & 0.6 \end{pmatrix}, \quad \boldsymbol{\Sigma}_0 = \begin{pmatrix} 0.07 & 0.05 \\ 0.05 & 0.07 \end{pmatrix}, \quad \boldsymbol{\Sigma}_{\check{\boldsymbol{\eta}}} = \begin{pmatrix} 0.079 & 0.052 \\ 0.052 & 0.100 \end{pmatrix}.$$

Eigenvalues of $\boldsymbol{\Phi}_0$ in this design are given by $0.5 \pm 0.070711i$ and the population R^2_Δ values are given by $R^2_{\Delta 2} = 0.23434$ and $R^2_{\Delta 2} = 0.23182$.

The parameter matrix $\boldsymbol{\Phi}_0$ was chosen such that the population R^2_Δ are comparable between Designs 1 and 2, but the extensibility condition is violated.

In Designs 3-4 we study finite sample properties of the estimators when the initial condition is not effect-stationary[26].

Design 3 (Stable PVAR with $\rho(\boldsymbol{\Phi}_0) = 0.50498$)**.** We take $\boldsymbol{\Phi}_0$ and $\boldsymbol{\Sigma}_0$ from Design 2, but with:

$$\boldsymbol{E}_i = 0.5 \times \boldsymbol{I}_2, \quad \boldsymbol{C}_i = \boldsymbol{I}_2, \quad \forall i = 1, \ldots, N.$$

$$\boldsymbol{\Sigma}_{\check{\boldsymbol{\eta}}, T=3} = \begin{pmatrix} 0.090 & 0.059 \\ 0.059 & 0.144 \end{pmatrix}, \quad \boldsymbol{\Sigma}_{\check{\boldsymbol{\eta}}, T=6} = \begin{pmatrix} 0.083 & 0.055 \\ 0.055 & 0.122 \end{pmatrix}.$$

Design 4 (Stable PVAR with $\rho(\boldsymbol{\Phi}_0) = 0.50498$)**.** We take $\boldsymbol{\Phi}_0$ and $\boldsymbol{\Sigma}_0$ from Design 2, but with:

$$\boldsymbol{E}_i = 1.5 \times \boldsymbol{I}_2, \quad \boldsymbol{C}_i = \boldsymbol{I}_2, \quad \forall i = 1, \ldots, N.$$

$$\boldsymbol{\Sigma}_{\check{\boldsymbol{\eta}}, T=3} = \begin{pmatrix} 0.069 & 0.045 \\ 0.045 & 0.074 \end{pmatrix}, \quad \boldsymbol{\Sigma}_{\check{\boldsymbol{\eta}}, T=6} = \begin{pmatrix} 0.074 & 0.049 \\ 0.049 & 0.083 \end{pmatrix}.$$

In Section 4.1 we presented theoretical results for the TML estimator when unrestricted cross sectional heteroscedasticity is present. This design is used to investigate the impact of *multiplicative* cross-sectional heteroscedasticity on the estimators.

Design 5 (Stable PVAR with $\rho(\boldsymbol{\Phi}_0) = 0.50498$ with non-i.i.d. $\boldsymbol{\varepsilon}_{i,t}$)**.** As a basis for this design we take $\boldsymbol{\Phi}_0$ and $\boldsymbol{\Sigma}_0$ from Design 1, but with:

$$\boldsymbol{E}_i = \boldsymbol{I}_2, \quad \boldsymbol{C}_i = \varphi_i \boldsymbol{I}_2, \quad \boldsymbol{\Sigma}_0(i) = \varphi_i^2 \boldsymbol{\Sigma}_0, \quad \varphi_i^2 \overset{iid}{\sim} \chi^2(1), \forall i = 1, \ldots, N.$$

[26]Note that effect non-stationarity in these designs has no impact on the first *unconditional* moment of the $\{\boldsymbol{y}_{i,t}\}_{t=0}^T$ process. It can be explained by the fact that $\mathrm{E}[\boldsymbol{\mu}_i] = \boldsymbol{0}_2$ is a sufficient condition for the $\{\boldsymbol{y}_{i,t}\}_{t=0}^T$ process to have a zero mean. Thus there is no reason to allow for mean non-stationarity by including $\boldsymbol{\gamma}$ parameter into the log-likelihood function, but it is crucial to allow for a covariance non-stationary initial condition.

The last design is dedicated to reveal the robustness properties of the TML estimator when time series heteroscedasticity is present. From Section 3.2 this estimator is *not* fixed T consistent in this case.

Design 6 (Stable PVAR with smooth time-series heteroscedasticity)**.** As a basis for this design we take $\boldsymbol{\Phi}_0$ and $\boldsymbol{\Sigma}$ from Design 1 $\boldsymbol{E}_i = \boldsymbol{C}_i = \boldsymbol{I}_2$, but with $\boldsymbol{\Sigma}_0(t)$ are generated as:

$$\boldsymbol{\Sigma}_0(t) = (0.95 - 0.05T + 0.1t) \times \boldsymbol{\Sigma}_0, \quad \forall t = 1, \ldots, T.$$

Particular form of the time series heteroscedasticity was chosen such that the $\bar{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}_0$. Analysis in Designs 5 and 6 can be extended by allowing *non-multiplicative* heteroscedasticity, however such analysis is beyond the scope of this paper.

For convenience we have multiplied both the mean and the median bias by *100*. Similarly to BHP we only present results for ϕ_{11} and ϕ_{12}, as results for the other two parameters are similar both quantitatively and qualitatively. The number of Monte Carlo simulations is set to $B = 10000$.

5.3. Technical Remarks

As starting values for TMLE estimation algorithm we used estimators available in a closed form. Namely, we used "AB-GMM", "Sys-GMM" and FDLS, the additive bias-corrected FE estimator as in Kiviet (1995) and the bias-corrected estimator of Hahn and Kuersteiner (2002). Here 'AB-GMM" stands for the Arellano and Bond (1991) estimator; "Sys-GMM" is the System estimator of Blundell and Bond (1998) which incorporates moment conditions based on the initial condition. All aforementioned GMM estimators are implemented in two steps, with the usual clustered weighting matrix used in the second step[27].

We denote by "TMLEr", the estimator which is obtained similarly as "TMLE", but instead of selecting the global maximum the local maximum that satisfies $|\hat{\boldsymbol{\Theta}} - \hat{\boldsymbol{\Sigma}}| \geq 0$ restriction is selected when possible[28] and global maximum otherwise. The TML estimator with imposed covariance stationarity is denoted by "TMLEc". Finally, we denote by "TMLEs" the estimator that is obtained by choosing the local maximum of TMLE objective function with the lowest spectral norm[29]. This choice is motivated by the fact that for univariate three-wave panel the second mode is always larger than the true mode; in PVAR one can think of spectral norm as measure of distance.

Regarding inference, for all the TML estimators we present results based on robust "sandwich" type standard errors labeled *(r)*. In case of GMM estimators, we provide rejection frequencies based on commonly used Windmeijer (2005) corrected S.E.

5.4. Results

5.4.1. Estimation

In this section we will briefly summarize the main findings of the MC study as presented in Tables B.1 to B.6. Inference related issues are discussed in the next section.

[27] That takes the form "$\boldsymbol{Z}'\boldsymbol{u}\boldsymbol{u}'\boldsymbol{Z}$".

[28] In principle this restriction is necessary but not sufficient for $\hat{\boldsymbol{\Theta}} - \hat{\boldsymbol{\Sigma}}$ to be p.s.d. However, for the purpose of exposition in this paper we will stick to this condition rather than checking non-negativity of the corresponding eigenvalues.

[29] However, unlike the univariate studies of Hsiao et al. (2002) and Hayakawa and Pesaran (2012), where the ϕ parameter was restricted to lie in the stationary region, in the numerical routine for the TMLE no restrictions on the parameter space are imposed.

Design 1. For the GMM based estimators we found very similar results as in the original study of BHP. Irrespective of N, the properties of all GMM estimators deteriorate as T and/or π increase and these effects are substantial both for diagonal and off-diagonal elements of $\boldsymbol{\Phi}$. Similarly, we can see that for small values of T, the performance of the TML estimator is directly related to the corresponding bias and the RMSE properties of the GMM estimators[30]. Hence using the estimators that are biased towards pseudo-true value , helps to find the second mode that happens to be the global maximum in that replication. On the other hand, if the resulting estimators (TMLEs, TMLEr, TMLEc) are restricted in some way, the strong dependence on starting values is no longer present (especially for TMLEs). In terms of both the bias and the RMSE we can see that the TMLEc estimator performs remarkably well irrespective of design parameter values for both diagonal and off-diagonal elements. The FDLS estimator does perform marginally worse as compared to the TMLEc estimator but still outperforms all the GMM estimators. All the TML estimators (except for TMLEc) tend to have an asymmetric finite sample distribution that results in corresponding discrepancies between estimates of mean and median.
In Section 3.3.3 we have mentioned that the second mode of the unrestricted TML estimator is located at $\boldsymbol{\Phi} = \boldsymbol{I}_m$. Based on the results in Table B.1 we can see that the diagonal elements for the TML estimator are *positively* biased towards 1, while the off-diagonal elements are *negatively* biased in direction of 0 (at least for small N and T). Thus the bimodality problem remains a substantial issue even for $T > 2$ and choosing global optimum is not always the best strategy as TMLEs clearly dominates TMLE for small values of T. For $T = 6$ the TMLEr and TMLEs provide equivalent results and some improvements over "global" standard TMLE.

Design 2. One of the implications of this setup is that the FDLS estimator is not consistent. More importantly, for this setup we do not know whether the bimodality issue for $T = 2$ is still present, thus the need for the TMLEr and TMLEs estimators is less obvious. However, the motivation becomes clear once we look at the corresponding results in Table B.2. TMLEs and TMLEr dominate TMLE in all cases, with TMLEs being the preferred choice. We can observe that the bias of the TML estimator in terms of both the magnitude and the sign does not change dramatically as compared to Design 1. Observe that the bias of the TMLEc in the diagonal elements does not decrease with T fast enough to match the performance of the TMLEr/TMLEs estimators. While for the off-diagonal elements quite a substantial bias remains even for $N = 250, T = 6$[31].

Designs 3 and 4. As it was expected, the properties of Sys-GMM (that rely on the effect-stationarity implied moment conditions) deteriorate significantly as compared to Design 2. We observe that for $\pi = 1$ the AB-GMM estimator is more biased in comparison to Design 2 (for Design 3), but is less biased than $\pi = 3$. The intuition of these patterns is similar to the one presented by Hayakawa (2009) within the univariate setting. Unlike the previous designs, the TML estimator exhibits lower bias for $\pi = 3$ despite the fact that the quality of the starting values diminished in the same way as in the effect-stationary case. Magnitudes of the effect non-stationary initial conditions considered in these designs are sufficient to ensure that the restrictions imposed from TMLEr estimator are

[30]This contrasts sharply with the finite sample results presented in BHP.

[31]As it will turn out later, these properties will play a major role to explain the finite sample properties of the LR test of covariance stationarity, that is presented in Online Appendix.

satisfied even for small values of N and T. As a result, the discrepancy between the nominal and the actual number of MC replications is very small.

Design 5. Unlike in Designs 3-4, the setup of Design 5 has no impact on consistency of most estimators analyzed (except FDLS). As can be clearly seen from Table B.6, the same can not be said about the finite sample properties. The introduction of cross-sectional variation in $\boldsymbol{\Sigma}_0(i)$ affected all estimation techniques by means of higher RMSE/MAE values. Unsurprisingly, these effects diminish as N increases, but are still present even for $N = 250$. On the other hand, the effect of cross-sectional variation is less clear for bias measures, with improvements for some estimators and higher bias for others.

Design 6. In this setup TML estimator is inconsistent due to time-series heteroscedasticity, the TMLEc estimator seems to be affected the most in terms of both the bias and precision. By comparing the results in Tables B.2 and B.6 we can conclude that fixed T inconsistency translates into more pronounced finite sample bias of the TML estimator. However, that is only true for diagonal elements (ϕ_{11} in this case) as the estimation quality of the off-diagonal elements remains unaffected. For $T = 6$ the bias of TMLE/TMLEs/TMLEr estimators diminishes, as can be expected given consistency in the large N, T framework. Furthermore, the Sys-GMM estimator, albeit still consistent, also shows some signs of deteriorating finite sample properties.

5.4.2. Size and Power properties

For brevity in this section we present the empirical rejection frequencies for ϕ_{11} only. Results for the other entries are available from the author upon request.
Below we will briefly summarize the main findings regarding the size and the power of the two-sided t test for different designs.

- Except for TMLEc, for $N = 100$ all estimators result in substantially oversized test statistics with relatively low power. In many cases rejection frequencies for alternatives close to the unit circle are of similar magnitude to size.

- When the estimator is consistent, the inference based on TMLEc serves as a benchmark both for size and power.

- In designs with the effect stationary initial condition (except $N = 250, T = 6$ to be discussed next), the empirical rejection frequencies based on all the TML (except for TMLEc) as well as the AB-GMM estimators do not result in symmetric power curves, due to the substantial finite sample bias of the estimators.

- Results for $T = 6$ and $N = 250$ suggest that the TML estimators without imposed stationarity restrictions are well sized and have good power properties in all designs with almost perfectly symmetric power curves.

- Although all the TML estimators (without imposed stationarity restriction) are inconsistent with time-series heteroscedastic error terms, the actual rejection frequencies for $N = 250$ are only marginally worse in comparison to the benchmark case. The same, however, can not be said about the TMLEc estimator.

- In design with cross-sectional heteroscedasticity, the TML based test statistics become more oversized, compared to the benchmark case. The only exception is the case with $N = 250$ and $T = 6$ where the actual size increases by at most 1%.

The results presented here suggest that under the assumption of time homoscedasticity, likelihood based techniques might serve as a viable alternative to the GMM based methods in the simple PVAR(1) model. Particularly, the TML estimator of BHP tends to be robust with respect to non-stationarity of the initial condition and cross-sectional heterogeneity of parameters. Furthermore, in finite sample likelihood based methods are robust even if smooth time-series heteroscedasticity is present. However, the TML estimator does suffer from serious bimodality problems when the number of cross-sectional units is small and the length of time series is short. In these cases the resulting estimator heavily depends on the way the estimator is chosen. As we have mentioned, in some designs in $30\% - 40\%$ of all MC replications no local maxima satisfying $|\hat{\boldsymbol{\Theta}} - \hat{\boldsymbol{\Sigma}}| > 0$ was available even for $N = 250$. However, this problem becomes marginal once $T = 6$ where such fractions drop to $1\% - 10\%$. Based on these results we suggest that the resulting TMLE estimator is chosen such that (when possible) local maxima should satisfies p.s.d. $|\hat{\boldsymbol{\Theta}} - \hat{\boldsymbol{\Sigma}}| > 0$ restriction (TMLEr), and otherwise the solution with smaller spectral norm should be chosen (TMLEs).

6. Concluding remarks

In this paper we aim at providing a thorough analysis of the performance of fixed T consistent estimation techniques for PVARX(1) model based on observations in first differences. We have mostly emphasized the results and properties of the likelihood based method. Our main goal was to investigate the robustness of aforementioned methods with respect to possible non-stationarity of the initial condition and/or cross-sectional heteroscedasticity.

We have extended the approach of BHP with inclusion of strictly exogenous regressors and shown how to construct a concentrated likelihood function for the autoregressive parameter only. Furthermore, we have shown that the TML is a "quasi" maximum likelihood estimator in several aspects. First of all, we have argued that it remains consistent with limiting normal distribution even when cross-sectional heterogeneity of the variance parameters or initial condition is not mean stationarity.

The key finding of this paper is that in the three-wave panel the expected log-likelihood function of BHP in the univariate setting does not have the unique maximum at the true value. This result has been shown to be robust irrespective of initialization. Furthermore, we have provided a sufficient condition for this result to hold for PVAR(1) in the three-wave panel.

Finally, we have conducted an extensive MC study with the emphasis on designs where the set of standard assumptions about the stationarity and the cross-sectional homoscedasticity were violated. Results suggest that likelihood based inference techniques might serve as a feasible alternative to GMM based methods in a simple PVARX(1) model. However, for small values of N and/or T the TML estimator is vulnerable to the choice of the starting values for the numerical optimization algorithm. These finite sample findings have been related to the bimodality results derived earlier in this paper. We proposed several ways of choosing the estimator among local maxima. Particularly, we suggest that the resulting TMLE estimator is chosen such that local maxima should satisfies p.s.d. restriction (TMLEr), and otherwise the solution with smaller spectral norm should be chosen (TMLEs).

References

Abadir, K. M. and J. R. Magnus (2002): "Notation in Econometrics: A Proposal for a Standard," *Econometrics Journal*, 5, 76–90.

Ahn, S. C. and P. Schmidt (1995): "Efficient Estimation of Models for Dynamic Panel Data," *Journal of Econometrics*, 68, 5–27.

——— (1997): "Efficient Estimation of Dynamic Panel Data Models: Alternative Assumptions and Simplified Estimation," *Journal of Econometrics*, 76, 309–321.

Ahn, S. C. and G. M. Thomas (2006): "Likelihood Based Inference for Dynamic Panel Data Models," Unpublished Manuscript.

Alonso-Borrego, C. and M. Arellano (1999): "Symmetrically Normalized Instrumental-Variable Estimation using Panel Data," *Journal of Business & Economic Statistics*, 17, 36–49.

Alvarez, J. and M. Arellano (2003): "The Time Series and Cross-Section Asymptotics of Dynamic Panel Data Estimators," *Econometrica*, 71(4), 1121–1159.

Amemiya, T. (1986): *Advanced Econometrics*, Wiley-Blackwell.

Arellano, M. (2003a): "Modeling Optimal Instrumental Variables for Dynamic Panel Data Models," Unpublished manuscript.

——— (2003b): *Panel Data Econometrics*, Advanced Texts in Econometrics, Oxford University Press.

Arellano, M. and S. Bond (1991): "Some Tests of Specification for Panel Data: Monte Carlo Evidence and an Application to Employment Equations," *Review of Economic Studies*, 58, 277–297.

Arellano, M. and O. Bover (1995): "Another Look at the Instrumental Variable Estimation of Error-components Models," *Journal of Econometrics*, 68, 29–51.

Binder, M., C. Hsiao, and M. H. Pesaran (2005): "Estimation and Inference in Short Panel Vector Autoregressions with Unit Root and Cointegration," *Econometric Theory*, 21, 795–837.

Blundell, R. W. and S. Bond (1998): "Initial Conditions and Moment Restrictions in Dynamic Panel Data Models," *Journal of Econometrics*, 87, 115–143.

Bond, S., C. Nauges, and F. Windmeijer (2005): "Unit Roots: Identification and Testing in Micro Panels," Working paper.

Bun, M. J. G. and M. A. Carree (2005): "Bias-Corrected Estimation in Dynamic Panel Data Models," *Journal of Business & Economic Statistics*, 23(2), 200–210.

Bun, M. J. G. and J. F. Kiviet (2006): "The Effects of Dynamic Feedbacks on LS and MM Estimator Accuracy in Panel Data Models," *Journal of Econometrics*, 132, 409–444.

Bun, M. J. G. and F. Windmeijer (2010): "The Weak Instrument Problem of the System GMM Estimator in Dynamic Panel Data Models," *The Econometrics Journal*, 13, 95–126.

Cao, B. and Y. Sun (2011): "Asymptotic Distributions of Impulse Response Functions in Short Panel Vector Autoregressions," *Journal of Econometrics*, 163, 127–143.

Ericsson, J. and M. Irandoust (2004): "The Productivity-bias Hypothesis and the PPP Theorem: New Evidence from Panel Vector Autoregressive Models," *Japan and the World Economy*, 16, 121–138.

Hahn, J. and G. Kuersteiner (2002): "Asymptotically Unbiased Inference for a Dynamic Panel Model with Fixed Effects When Both N and T are Large," *Econometrica*, 70(4), 1639–1657.

Han, C. and P. C. B. Phillips (2010): "GMM Estimation for Dynamic Panels with Fixed Effects and Strong Instruments at Unity," *Econometric Theory*, 26, 119–151.

——— (2013): "First difference maximum likelihood and dynamic panel estimation," *Journal of Econometrics*, 175, 35–45.

Hayakawa, K. (2007): "Consistent OLS Estimation of AR(1) Dynamic Panel Data Models with Short Time Series," *Applied Economics Letters*, 14:15, 1141–1145.

——— (2009): "On the Effect of Mean-Nonstationarity in Dynamic Panel Data Models," *Journal of Econometrics*, 153, 133–135.

——— (2013): "An Improved GMM Estimation of Panel VAR Models," Working paper.

Hayakawa, K. and M. H. Pesaran (2012): "Robust Standard Errors in Transformed Likelihood Estimation of Dynamic Panel Data Models," Working Paper.

Holtz-Eakin, D., W. K. Newey, and H. S. Rosen (1988): "Estimating Vector Autoregressions with Panel Data," *Econometrica*, 56, 1371–1395.

Hsiao, C., M. H. Pesaran, and A. K. Tahmiscioglu (2002): "Maximum Likelihood Estimation of Fixed Effects Dynamic Panel Data Models Covering Short Time Periods," *Journal of Econometrics*, 109, 107–150.

Juessen, F. and L. Linnemann (2010): "Estimating Panel VARs from Macroeconomic Data: Some Monte Carlo Evidence and an Application to OECD Public Spending Shocks," SFB Discussion Paper 823.

Juodis, A. (2013): "A note on bias-corrected estimation in dynamic panel data models," *Economics Letters*, 118, 435–438.

——— (2014): "Supplement to "First Difference Transformation in Panel VAR models: Robustness, Estimation and Inference"." `http://arturas.economists.lt/FD_online.pdf`.

Kiviet, J. F. (1995): "On Bias, Inconsistency, and Efficiency of Various Estimators in Dynamic Panel Data Models," *Journal of Econometrics*, 68, 53–78.

——— (2007): "Judging Contending Estimators by Simulation: Tournaments in Dynamic Panel Data Models," in *The Refinement of Econometric Estimation and Test Procedures*, ed. by G. Phillips and E. Tzavalis, Cambridge University Press, chap. 11, 282–318.

Koutsomanoli-Filippaki, A. and E. Mamatzakis (2009): "Performance and Merton-type Default Risk of Listed Banks in the EU: A Panel VAR Approach," *Journal of Banking and Finance*, 33, 2050–2061.

Kruiniger, H. (2002): "On the Estimation of Panel Regression Models with Fixed Effects," Working paper 450, Queen Mary, University of London.

——— (2007): "An Efficient Linear GMM Estimator for the Covariance Stationary Ar(1)/Unit Root Model for Panel Data," *Econometric Theory*, 23, 519–535.

——— (2008): "Maximum Likelihood Estimation and Inference Methods for the Covariance Stationary Panel AR(1)/Unit Root Model," *Journal of Econometrics*, 144, 447–464.

——— (2013): "Quasi ML estimation of the panel AR(1) model with arbitrary initial conditions," *Journal of Econometrics*, 173, 175–188.

Maddala, G. (1971): "The Use of Variance Components Models in Pooling Cross Section and Time Series Data," *Econometrica*, 39, 341–35.

Magnus, J. R. and H. Neudecker (2007): *Matrix Differential Calculus with Applications in Statistics and Econometrics*, John Wiley & Sons.

Molinari, L. G. (2008): "Determinants of Block Tridiagonal Matrices," *Linear Algebra and its Applications*, 429, 2221–2226.

Mutl, J. (2009): "Panel VAR Models with Spatial Dependence," Working Paper.

Nickell, S. (1981): "Biases in Dynamic Models with Fixed Effects," *Econometrica*, 49, 1417–1426.

Ramalho, J. J. S. (2005): "Feasible bias-corrected OLS, within-groups, and first-differences estimators for typical micro and macro AR(1) panel data models," *Empirical Economics*, 30, 735–748.

White, H. (2000): *Asymptotic Theory for Econometricians*, Economic Theory, Econometrics, and Mathematical Economics, Academic Press, 2 ed.

Windmeijer, F. (2005): "A Finite Sample Correction for the Variance of Linear Efficient Two-Step GMM Estimators," *Journal of Econometrics*, 126, 25–51.

Appendices

Appendix A. Proofs

Appendix A.1. TMLE estimator

Firstly, we define a set of new auxiliary variables, which will be handy during the derivations of differentials:

$$\boldsymbol{Z}_N(\boldsymbol{\kappa}) := \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})', \quad \boldsymbol{Q}_N(\boldsymbol{\kappa}) := \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{\boldsymbol{y}}_{i,t-1}(\tilde{\boldsymbol{y}}_{i,t} - \boldsymbol{\Phi}\tilde{\boldsymbol{y}}_{i,t-1})',$$

$$\boldsymbol{M}_N(\boldsymbol{\kappa}) := \frac{T}{N}\sum_{i=1}^{N}(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}\ddot{\boldsymbol{y}}_{i-})(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}\ddot{\boldsymbol{y}}_{i-})', \quad \boldsymbol{N}_N(\boldsymbol{\kappa}) := \frac{T}{N}\sum_{i=1}^{N}\ddot{\boldsymbol{y}}_{i-}(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}\ddot{\boldsymbol{y}}_{i-})',$$

$$\boldsymbol{R}_N := \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{\boldsymbol{y}}_{i,t-1}\tilde{\boldsymbol{y}}'_{i,t-1}, \quad \boldsymbol{P}_N := \frac{T}{N}\sum_{i=1}^{N}\ddot{\boldsymbol{y}}_{i-}\ddot{\boldsymbol{y}}'_{i-}, \boldsymbol{\Xi} := \sum_{l=0}^{T-2}(T-1-l)\boldsymbol{\Phi}_0^l.$$

In the derivations below we will use several results concerning differentials (for more details refer to Magnus and Neudecker (2007)):

$$\begin{aligned} \mathrm{dlog}\,|\boldsymbol{X}| &= \mathrm{tr}\,(\boldsymbol{X}^{-1}(\mathrm{d}\boldsymbol{X})), \quad \mathrm{d}(\mathrm{tr}\,\boldsymbol{X}) = \mathrm{tr}\,(\mathrm{d}\boldsymbol{X}), \\ \mathrm{d}(\mathrm{vec}\,\boldsymbol{X}) &= \mathrm{vec}\,(\mathrm{d}\boldsymbol{X}), \quad \mathrm{d}\boldsymbol{X}^{-1} = -\boldsymbol{X}^{-1}(\mathrm{d}\boldsymbol{X})\boldsymbol{X}^{-1}, \\ \mathrm{d}\boldsymbol{X}\boldsymbol{Y} &= (\mathrm{d}\boldsymbol{X})\boldsymbol{Y} + \boldsymbol{X}(\mathrm{d}\boldsymbol{Y}), \quad \mathrm{d}(\boldsymbol{X}\otimes\boldsymbol{X}) = \mathrm{d}(\boldsymbol{X})\otimes\boldsymbol{X} + \boldsymbol{X}\otimes\mathrm{d}(\boldsymbol{X}), \\ \mathrm{vec}\,(\mathrm{d}\boldsymbol{X}\otimes\boldsymbol{X}) &= (\boldsymbol{I}_m\otimes\mathbf{K}_m\otimes\boldsymbol{I}_m)(\boldsymbol{I}_{m^2}\otimes\mathrm{vec}\,\boldsymbol{X})\,\mathrm{vec}\,\mathrm{d}(\boldsymbol{X}) \end{aligned}$$

Appendix A.1.1. Auxiliary results

Lemma Appendix A.1.

$$\boldsymbol{\Upsilon} := \sum_{l=0}^{T-1}\boldsymbol{\Phi}_0^l - T\boldsymbol{I}_m + \left(\sum_{l=0}^{T-2}(T-l)\boldsymbol{\Phi}_0^l - \sum_{l=0}^{T-2}\boldsymbol{\Phi}_0^l\right)(\boldsymbol{I}_m - \boldsymbol{\Phi}_0) = \mathbf{O}_m$$

Proof.

$$\begin{aligned} \boldsymbol{\Upsilon} &:= \sum_{l=0}^{T-1}\boldsymbol{\Phi}_0^l - T\boldsymbol{I}_m + \left(\sum_{l=0}^{T-2}(T-l)\boldsymbol{\Phi}_0^l - \sum_{l=0}^{T-2}\boldsymbol{\Phi}_0^l\right)(\boldsymbol{I}_m - \boldsymbol{\Phi}_0) \\ &= \boldsymbol{\Phi}_0^{T-1} + \sum_{l=0}^{T-2}\boldsymbol{\Phi}_0^{l+1} - T\boldsymbol{I}_m + T\left(\sum_{l=0}^{T-2}\boldsymbol{\Phi}_0^l - \sum_{l=1}^{T-1}\boldsymbol{\Phi}_0^l\right) - \left(\sum_{l=1}^{T-2}l\boldsymbol{\Phi}_0^l - \sum_{l=1}^{T-1}(l-1)\boldsymbol{\Phi}_0^l\right) \\ &= \boldsymbol{\Phi}_0^{T-1} + \sum_{l=1}^{T-1}\boldsymbol{\Phi}_0^l - T\boldsymbol{I}_m + T(\boldsymbol{I}_m - \boldsymbol{\Phi}_0^{T-1}) - \left(\sum_{l=1}^{T-2}\boldsymbol{\Phi}_0^l - (T-2)\boldsymbol{\Phi}_0^{T-1}\right) \\ &= \boldsymbol{\Phi}_0^{T-1} + \sum_{l=0}^{T-2}\boldsymbol{\Phi}_0^{l+1} - T\boldsymbol{\Phi}_0^{T-1} - \left(\sum_{l=1}^{T-2}\boldsymbol{\Phi}_0^l - (T-2)\boldsymbol{\Phi}_0^{T-1}\right) \\ &= (1-T)\boldsymbol{\Phi}_0^{T-1} + \boldsymbol{\Phi}_0^{T-1} + (T-2)\boldsymbol{\Phi}_0^{T-1} = \mathbf{O}_m. \end{aligned}$$

□

Lemma Appendix A.2. *Under Assumptions **SA*** the following relation holds:*

$$\mathrm{E}\left[\boldsymbol{N}_N(\boldsymbol{\kappa}_0)\right] = \frac{1}{T}\boldsymbol{\Xi}\boldsymbol{\Theta}_0.$$

for $\boldsymbol{\Theta}_0 = \boldsymbol{\Sigma}_0 + T(\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\boldsymbol{E}[\boldsymbol{u}_{i,0}\boldsymbol{u}_{i,0}'](\boldsymbol{I}_m - \boldsymbol{\Phi}_0)'$.

Proof.

$$\begin{aligned}
&\mathrm{E}\left[\boldsymbol{N}_N(\boldsymbol{\kappa}_0)'\right] = \mathrm{E}\left(\frac{T}{N}\sum_{i=1}^{N}(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}_0\ddot{\boldsymbol{y}}_{i-})\ddot{\boldsymbol{y}}_{i-}'\right)\\
&= \mathrm{E}\left[(-(\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\boldsymbol{u}_{i,0} + \bar{\boldsymbol{\varepsilon}}_i)\left(\left(\sum_{s=0}^{T-1}\boldsymbol{\Phi}_0^s - T\boldsymbol{I}_m\right)\boldsymbol{y}_{i,0} + \left(\sum_{l=0}^{T-2}(T-1-l)\boldsymbol{\Phi}_0^l\right)(\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\boldsymbol{\mu}_i + \left(\sum_{t=1}^{T-1}\sum_{s=0}^{t-1}\boldsymbol{\Phi}_0^s\boldsymbol{\varepsilon}_{i,t-s}\right)\right)\right.\\
&= \mathrm{E}\left[(-(\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\boldsymbol{u}_{i,0} + \bar{\boldsymbol{\varepsilon}}_i)\left(\boldsymbol{\Upsilon}\boldsymbol{y}_{i,0} - \left(\sum_{l=0}^{T-2}(T-1-l)\boldsymbol{\Phi}_0^l\right)(\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\boldsymbol{u}_{i,0} + \left(\sum_{t=1}^{T-1}\sum_{s=0}^{t-1}\boldsymbol{\Phi}_0^s\boldsymbol{\varepsilon}_{i,t-s}\right)\right)'\right].
\end{aligned}$$

In Lemma Appendix A.1 we showed that $\boldsymbol{\Upsilon} = \mathbf{O}_m$, hence it follows that:

$$\begin{aligned}
\mathrm{E}\left[\frac{T}{N}\sum_{i=1}^{N}(\ddot{\boldsymbol{y}}_i - \boldsymbol{\Phi}_0\ddot{\boldsymbol{y}}_{i-})\ddot{\boldsymbol{y}}_{i-}'\right] &= \mathrm{E}\left[(-(\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\boldsymbol{u}_{i,0} + \bar{\boldsymbol{\varepsilon}}_i)\left(-\boldsymbol{\Xi}(\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\boldsymbol{u}_{i,0} + \left(\sum_{t=1}^{T-1}\sum_{s=0}^{t-1}\boldsymbol{\Phi}_0^s\boldsymbol{\varepsilon}_{i,t-s}\right)\right)'\right]\\
&= (\boldsymbol{I}_m - \boldsymbol{\Phi}_0)\boldsymbol{E}[\boldsymbol{u}_{i,0}\boldsymbol{u}_{i,0}'](\boldsymbol{I}_m - \boldsymbol{\Phi}_0)'\boldsymbol{\Xi}' + \frac{1}{T}\boldsymbol{\Sigma}_0\boldsymbol{\Xi}' = \frac{1}{T}\boldsymbol{\Theta}_0\boldsymbol{\Xi}'.
\end{aligned}$$

□

Appendix A.1.2. Log-likelihood function

Theorem 3.1. Let:

$$\Delta\boldsymbol{\tau}_i = \begin{pmatrix} \Delta\boldsymbol{y}_{i,1} \\ \Delta\boldsymbol{\varepsilon}_{i,2} \\ \vdots \\ \Delta\boldsymbol{\varepsilon}_{i,T} \end{pmatrix}_{mT\times 1}, \quad \boldsymbol{C}_T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix}_{T\times T}, \boldsymbol{L}_T = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}_{T\times T}$$

and let $\boldsymbol{D}$ be a $[T \times T+1]$ matrix which transforms a $[T+1 \times 1]$ vector x into a $[T \times 1]$ vector of corresponding first differences. Also define $\boldsymbol{\Theta} := T(\boldsymbol{\Psi} - \boldsymbol{\Sigma}) + \boldsymbol{\Sigma}$ and $\boldsymbol{\Omega} := \boldsymbol{\Sigma}^{-1}\boldsymbol{\Theta}$. If we denote $\boldsymbol{\Gamma} := \boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi}$ it then follows that:

$$\begin{aligned}
\boldsymbol{\Sigma}_{\Delta\boldsymbol{\tau}} &= (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma})\begin{pmatrix} \boldsymbol{\Gamma} & -\boldsymbol{I}_m & \mathbf{O}_m & \cdots & \mathbf{O}_m \\ -\boldsymbol{I}_m & 2\boldsymbol{I}_m & \ddots & \ddots & \vdots \\ \mathbf{O}_m & \ddots & \ddots & \ddots & \mathbf{O}_m \\ \vdots & \ddots & \ddots & \ddots & -\boldsymbol{I}_m \\ \mathbf{O}_m & \cdots & \mathbf{O}_m & -\boldsymbol{I}_m & 2\boldsymbol{I}_m \end{pmatrix}\\
&= (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma})\left[(\boldsymbol{D}\boldsymbol{D}' \otimes \boldsymbol{I}_m) + (\boldsymbol{e}_1\boldsymbol{e}_1' \otimes (\boldsymbol{\Gamma} - 2\boldsymbol{I}_m))\right]\\
&= (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma})\left[((\boldsymbol{C}_T'\boldsymbol{C}_T)^{-1} \otimes \boldsymbol{I}_m) + (\boldsymbol{e}_1\boldsymbol{e}_1' \otimes (\boldsymbol{\Gamma} - \boldsymbol{I}_m))\right].
\end{aligned}$$

Subsequently the determinant is given by (using the fact that $|\boldsymbol{C}_T| = 1$):

$$\begin{aligned}
|\boldsymbol{\Sigma}_{\Delta\boldsymbol{\tau}}| &= |\boldsymbol{\Sigma}|^T |((\boldsymbol{C}_T'\boldsymbol{C}_T)^{-1} \otimes \boldsymbol{I}_m) + (\boldsymbol{e}_1\boldsymbol{e}_1' \otimes (\boldsymbol{\Gamma} - \boldsymbol{I}_m))| = |\boldsymbol{\Sigma}|^T |\boldsymbol{I}_m + (\boldsymbol{e}_1'\boldsymbol{C}_T'\boldsymbol{C}_T\boldsymbol{e}_1(\boldsymbol{\Gamma} - \boldsymbol{I}_m))||(\boldsymbol{C}_T'\boldsymbol{C}_T)^{-1}| \\
&= |\boldsymbol{\Sigma}|^T |\boldsymbol{I}_m + (\boldsymbol{e}_1'\boldsymbol{C}_T'\boldsymbol{C}_T\boldsymbol{e}_1(\boldsymbol{\Gamma} - \boldsymbol{I}_m))| = |\boldsymbol{\Sigma}|^T |\boldsymbol{I}_m + T(\boldsymbol{\Gamma} - \boldsymbol{I}_m)| = |\boldsymbol{\Sigma}|^T |\boldsymbol{\Omega}| = |\boldsymbol{\Sigma}|^{T-1}|\boldsymbol{\Theta}|,
\end{aligned}$$

where the second line follows by means of the Matrix Determinant Lemma[32]. Using the **Woodbury** formula we can evaluate $\boldsymbol{\Sigma}_{\Delta\boldsymbol{\tau}}^{-1}$:

$$\begin{aligned}
\boldsymbol{\Sigma}_{\Delta\boldsymbol{\tau}}^{-1} &= \left[((\boldsymbol{C}_T'\boldsymbol{C}_T)^{-1} \otimes \boldsymbol{I}_m) + (\boldsymbol{e}_1\boldsymbol{e}_1' \otimes (\boldsymbol{\Gamma} - \boldsymbol{I}_m))\right]^{-1} (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}^{-1}) \\
&= \left[((\boldsymbol{C}_T'\boldsymbol{C}_T) \otimes \boldsymbol{I}_m) - ((\boldsymbol{C}_T'\boldsymbol{C}_T\boldsymbol{e}_1) \otimes \boldsymbol{I}_m)\left((\boldsymbol{\Gamma} - \boldsymbol{I}_m)^{-1} + T\boldsymbol{I}_m\right)((\boldsymbol{e}_1'\boldsymbol{C}_T'\boldsymbol{C}_T) \otimes \boldsymbol{I}_m)\right](\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}^{-1}) \\
&= (\boldsymbol{C}_T' \otimes \boldsymbol{I}_m)\boldsymbol{U}(\boldsymbol{C}_T \otimes \boldsymbol{I}_m)(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}^{-1}) = (\boldsymbol{C}_T' \otimes \boldsymbol{I}_m)\boldsymbol{U}(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}^{-1})(\boldsymbol{C}_T \otimes \boldsymbol{I}_m),
\end{aligned}$$

where $\boldsymbol{U}$ is:

$$\begin{aligned}
\boldsymbol{U} &= \boldsymbol{I}_{Tm} - ((\boldsymbol{C}_T\boldsymbol{e}_1) \otimes \boldsymbol{I}_m)\left((\boldsymbol{\Gamma} - \boldsymbol{I}_m)\boldsymbol{\Omega}^{-1}\right)((\boldsymbol{e}_1'\boldsymbol{C}_T') \otimes \boldsymbol{I}_m) \\
&= \boldsymbol{I}_{Tm} - (\boldsymbol{\imath}_T \otimes \boldsymbol{I}_m)\left((\boldsymbol{\Gamma} - \boldsymbol{I}_m)\boldsymbol{\Omega}^{-1}\right)(\boldsymbol{\imath}_T' \otimes \boldsymbol{I}_m) = \boldsymbol{I}_{Tm} - \boldsymbol{\imath}_T\boldsymbol{\imath}_T' \otimes \left((\boldsymbol{\Gamma} - \boldsymbol{I}_m)\boldsymbol{\Omega}^{-1}\right) \\
&= \boldsymbol{I}_{Tm} - \frac{1}{T}\boldsymbol{\imath}_T\boldsymbol{\imath}_T' \otimes \left(\boldsymbol{I}_m - \boldsymbol{\Omega}^{-1}\right) = \boldsymbol{I}_{Tm} - \frac{1}{T}\boldsymbol{\imath}_T\boldsymbol{\imath}_T' \otimes \boldsymbol{I}_m + \frac{1}{T}\boldsymbol{\imath}_T\boldsymbol{\imath}_T' \otimes \boldsymbol{\Omega}^{-1} \\
&= \boldsymbol{W}_T \otimes \boldsymbol{I}_m + \frac{1}{T}\boldsymbol{\imath}_T\boldsymbol{\imath}_T' \otimes \boldsymbol{\Omega}^{-1},
\end{aligned}$$

so that:

$$\boldsymbol{\Sigma}_{\Delta\boldsymbol{\tau}}^{-1} = (\boldsymbol{C}_T' \otimes \boldsymbol{I}_m)\left[\boldsymbol{W}_T \otimes \boldsymbol{\Sigma}^{-1} + \frac{1}{T}\boldsymbol{\imath}_T\boldsymbol{\imath}_T' \otimes \boldsymbol{\Theta}^{-1}\right](\boldsymbol{C}_T \otimes \boldsymbol{I}_m).$$

Now using the fact that $\boldsymbol{R} = (\boldsymbol{I}_{Tm} - \boldsymbol{L}_T \otimes \boldsymbol{\Phi})$ and defining $\boldsymbol{z}_i = (\boldsymbol{y}_{i,0}, \ldots, \boldsymbol{y}_{i,T})$:

$$\begin{aligned}
\boldsymbol{Z} &:= (\boldsymbol{C}_T \otimes \boldsymbol{I}_m)(\boldsymbol{I}_{Tm} - \boldsymbol{L}_T \otimes \boldsymbol{\Phi})\,\mathrm{vec}\,(\boldsymbol{z}_i\boldsymbol{D}') \\
&= \mathrm{vec}\,(\boldsymbol{z}_i\boldsymbol{D}'\boldsymbol{C}_T' - \boldsymbol{\Phi}\boldsymbol{z}_i\boldsymbol{D}'\boldsymbol{L}_T'\boldsymbol{C}_T') = \mathrm{vec}\,((\boldsymbol{C}_T\boldsymbol{D}\boldsymbol{z}_i')' - \boldsymbol{\Phi}(\boldsymbol{C}_T\boldsymbol{L}_T\boldsymbol{D}\boldsymbol{z}_i')') \\
&= \mathrm{vec}\,((\boldsymbol{Y}_i - \boldsymbol{\imath}_T\boldsymbol{y}_{i,0})' - \boldsymbol{\Phi}(\boldsymbol{Y}_{i-} - \boldsymbol{\imath}_T\boldsymbol{y}_{i,0})').
\end{aligned}$$

Hence the log likelihood function of BHP can be rewritten in the following way (where $\boldsymbol{\kappa} = (\boldsymbol{\phi}', \boldsymbol{\sigma}', \boldsymbol{\theta}')'$):

$$\ell(\boldsymbol{\kappa}) = c - \frac{N}{2}\left((T-1)\log|\boldsymbol{\Sigma}| + \log|\boldsymbol{\Theta}| + \mathrm{tr}\,(\boldsymbol{\Sigma}^{-1}\boldsymbol{Z}_N(\boldsymbol{\kappa})) + \mathrm{tr}\,(\boldsymbol{\Theta}^{-1}\boldsymbol{M}_N(\boldsymbol{\kappa}))\right). \qquad \text{(Appendix A.1)}$$

In order to include exogenous regressors in the model we denote the following quantities:

$$\boldsymbol{\gamma} = \boldsymbol{G}\Delta\boldsymbol{X}_i^{\dagger}, \quad \check{\boldsymbol{X}}_i = (\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,T}).$$

The $\boldsymbol{Z}$ term in this case is given by:

$$\begin{aligned}
\boldsymbol{Z} &:= (\boldsymbol{C}_T \otimes \boldsymbol{I}_m)\left((\boldsymbol{I}_{Tm} - \boldsymbol{L}_T \otimes \boldsymbol{\Phi})\,\mathrm{vec}\,(\boldsymbol{z}_i\boldsymbol{D}') - (\boldsymbol{I}_T \otimes \boldsymbol{B})\,\mathrm{vec}\,(\Delta\boldsymbol{X}_i) - \mathrm{vec}\,(\boldsymbol{\gamma}\boldsymbol{e}_1')\right) \\
&= \mathrm{vec}\,((\boldsymbol{Y}_i - \boldsymbol{\imath}_T(\boldsymbol{y}_{i,0} + \boldsymbol{\gamma}))' - \boldsymbol{\Phi}(\boldsymbol{Y}_{i-} - \boldsymbol{\imath}_T\boldsymbol{y}_{i,0})' - \boldsymbol{B}(\check{\boldsymbol{X}}_i - \boldsymbol{\imath}_T\boldsymbol{x}_{i,0})').
\end{aligned}$$

Result follows directly based on derivations for PVAR(1) model by redefining $\boldsymbol{Z}_N$ and $\boldsymbol{M}_N$. □

[32] Alternatively $|\boldsymbol{\Sigma}_{\Delta\boldsymbol{\tau}}|$ can be evaluated using the general formula for tridiagonal matrices in Molinari (2008).

Appendix A.1.3. Score vector

Lemma 3.4. Here for simplicity we derive first differential of $\ell(\boldsymbol{\kappa})$ without exogenous regressors:

$$\begin{aligned}
-\frac{2}{N}\,\mathrm{d}\ell(\boldsymbol{\kappa}) &= (T-1)\,\mathrm{tr}\,(\boldsymbol{\Sigma}^{-1}(\mathrm{d}\boldsymbol{\Sigma})) + \mathrm{tr}\,(\boldsymbol{\Theta}^{-1}(\mathrm{d}\boldsymbol{\Theta})) \\
&\quad - \mathrm{tr}\,(\boldsymbol{\Sigma}^{-1}(\mathrm{d}\boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1}\boldsymbol{Z}_N(\boldsymbol{\kappa})) - \mathrm{tr}\,(\boldsymbol{\Theta}^{-1}(\mathrm{d}\boldsymbol{\Theta})\boldsymbol{\Theta}^{-1}\boldsymbol{M}_N(\boldsymbol{\kappa})) \\
&\quad + \mathrm{tr}\,(\boldsymbol{\Sigma}^{-1}(\mathrm{d}\boldsymbol{Z}_N(\boldsymbol{\kappa}))) + \mathrm{tr}\,(\boldsymbol{\Theta}^{-1}(\mathrm{d}\boldsymbol{M}_N(\boldsymbol{\kappa}))) \\
&= \mathrm{tr}\,(\boldsymbol{\Sigma}^{-1}((T-1)\boldsymbol{\Sigma} - \boldsymbol{Z}_N(\boldsymbol{\kappa}))\boldsymbol{\Sigma}^{-1}(\mathrm{d}\boldsymbol{\Sigma})) + \mathrm{tr}\,(\boldsymbol{\Theta}^{-1}(\boldsymbol{\Theta} - \boldsymbol{M}_N(\boldsymbol{\kappa}))\boldsymbol{\Theta}^{-1}(\mathrm{d}\boldsymbol{\Theta})) \\
&\quad - 2\,\mathrm{tr}\left(\boldsymbol{\Sigma}^{-1}((\mathrm{d}\boldsymbol{\Phi})\boldsymbol{Q}_N(\boldsymbol{\kappa}))\right) - 2\,\mathrm{tr}\left(\boldsymbol{\Theta}^{-1}((\mathrm{d}\boldsymbol{\Phi})\boldsymbol{N}_N(\boldsymbol{\kappa}))\right).
\end{aligned}$$

Based on these derivations we conclude that the corresponding $[2m^2 + m \times 1]$ score vector is given by:

$$\nabla(\boldsymbol{\kappa}) = N\begin{pmatrix} \mathrm{vec}\,(\boldsymbol{\Sigma}^{-1}\boldsymbol{Q}_N(\boldsymbol{\kappa})' + \boldsymbol{\Theta}^{-1}\boldsymbol{N}_N(\boldsymbol{\kappa})') \\ \mathbf{D}_m'\,\mathrm{vec}\,(-\tfrac{1}{2}(\boldsymbol{\Sigma}^{-1}((T-1)\boldsymbol{\Sigma} - \boldsymbol{Z}_N(\boldsymbol{\kappa}))\boldsymbol{\Sigma}^{-1})) \\ \mathbf{D}_m'\,\mathrm{vec}\,(-\tfrac{1}{2}(\boldsymbol{\Theta}^{-1}(\boldsymbol{\Theta} - \boldsymbol{M}_N(\boldsymbol{\kappa}))\boldsymbol{\Theta}^{-1})) \end{pmatrix}. \qquad \text{(Appendix A.2)}$$

Zero mean result follows trivially from Lemma Appendix A.2 and the fact that $\mathrm{E}[\boldsymbol{\Sigma}_0^{-1}\boldsymbol{Q}_N(\boldsymbol{\kappa}_0)] = (1/T)\boldsymbol{\Xi}$ (the "Nickell bias"). □

Proposition 3.5. We need to derive the exact expression for $\mathrm{vec}\,\mathrm{d}\boldsymbol{\Theta}$ under assumption that $\mathrm{vec}\,\mathrm{E}[\boldsymbol{u}_{i,0}\boldsymbol{u}_{i,0}'] = (\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}\otimes\boldsymbol{\Phi})^{-1}\,\mathrm{vec}\,\boldsymbol{\Sigma}$. At first, we rewrite the expression for $\mathrm{vec}\,\boldsymbol{\Theta}$ (we prefer to work with $\mathrm{vec}\,(\cdot)$ rather than $\mathrm{vech}\,(\cdot)$ to avoid excessive use of duplication matrix $\mathbf{D}_m$):

$$\begin{aligned}
\mathrm{vec}\,\boldsymbol{\Theta} &= \mathrm{vec}\,\boldsymbol{\Sigma} + T\,((\boldsymbol{I}_m - \boldsymbol{\Phi})\otimes(\boldsymbol{I}_m - \boldsymbol{\Phi}))\,\mathrm{vec}\,\mathrm{E}[\boldsymbol{u}_{i,0}\boldsymbol{u}_{i,0}'] \\
&= \mathrm{vec}\,\boldsymbol{\Sigma} + T\,((\boldsymbol{I}_m - \boldsymbol{\Phi})\otimes(\boldsymbol{I}_m - \boldsymbol{\Phi}))\,(\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}\otimes\boldsymbol{\Phi})^{-1}\,\mathrm{vec}\,\boldsymbol{\Sigma} = \boldsymbol{S}_2\,\mathrm{vec}\,\boldsymbol{\Sigma}
\end{aligned}$$

Using rules for differentials we get that:

$$\mathrm{d}(\mathrm{vec}\,\boldsymbol{\Theta}) = \boldsymbol{S}_2\,\mathrm{d}(\mathrm{vec}\,\boldsymbol{\Sigma}) + \mathrm{d}(\boldsymbol{S}_2)\,\mathrm{vec}\,\boldsymbol{\Sigma}.$$

Using the product rule for differentials:

$$\begin{aligned}
\frac{1}{T}\,\mathrm{d}(\boldsymbol{S}_2) &= -\left(\mathrm{d}(\boldsymbol{\Phi})\otimes(\boldsymbol{I}_m - \boldsymbol{\Phi}) + (\boldsymbol{I}_m - \boldsymbol{\Phi})\otimes\mathrm{d}(\boldsymbol{\Phi})\right)(\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}\otimes\boldsymbol{\Phi})^{-1} \\
&\quad + ((\boldsymbol{I}_m - \boldsymbol{\Phi})\otimes(\boldsymbol{I}_m - \boldsymbol{\Phi}))\,(\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}\otimes\boldsymbol{\Phi})^{-1}\,(\mathrm{d}(\boldsymbol{\Phi})\otimes\boldsymbol{\Phi} + \boldsymbol{\Phi}\otimes\mathrm{d}(\boldsymbol{\Phi}))\,(\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}\otimes\boldsymbol{\Phi})^{-1}
\end{aligned}$$

Recall definition of $\mathrm{E}[\boldsymbol{u}_{i,0}\boldsymbol{u}_{i,0}'] = \boldsymbol{\Psi}_0$ and define $\boldsymbol{\psi}_0 := \mathrm{vec}\,\boldsymbol{\Psi}_0$. As $\mathrm{d}(\boldsymbol{S}_2)\,\mathrm{vec}\,\boldsymbol{\Sigma}$ is already a vector by taking $\mathrm{vec}\,(\cdot)$ of this term nothing changes:

$$\begin{aligned}
\frac{1}{T}\,\mathrm{vec}\,(\mathrm{d}(\boldsymbol{S}_2)\,\mathrm{vec}\,\boldsymbol{\Sigma}) &= -(\boldsymbol{\psi}_0'\otimes\boldsymbol{I}_{m^2})\,\mathrm{vec}\,(\mathrm{d}(\boldsymbol{\Phi})\otimes(\boldsymbol{I}_m - \boldsymbol{\Phi}) + (\boldsymbol{I}_m - \boldsymbol{\Phi})\otimes\mathrm{d}(\boldsymbol{\Phi})) \\
&\quad + (\boldsymbol{\psi}_0'\otimes\left(((\boldsymbol{I}_m - \boldsymbol{\Phi})\otimes(\boldsymbol{I}_m - \boldsymbol{\Phi}))\,(\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}\otimes\boldsymbol{\Phi})^{-1}\right))\,\mathrm{vec}\,(\mathrm{d}(\boldsymbol{\Phi})\otimes(\boldsymbol{\Phi}) + (\boldsymbol{\Phi})\otimes\mathrm{d}(\boldsymbol{\Phi}))
\end{aligned}$$

Using the formula for $\mathrm{vec}\,(\mathrm{d}\boldsymbol{X}\otimes\boldsymbol{X})$:

$$\begin{aligned}
\frac{1}{T}\,\mathrm{d}(\boldsymbol{S}_2)\,\mathrm{vec}\,\boldsymbol{\Sigma} &= -\,(\boldsymbol{\psi}_0'\otimes\boldsymbol{I}_{m^2})(\boldsymbol{I}_m\otimes\mathrm{K}_m\otimes\boldsymbol{I}_m)(\boldsymbol{I}_{m^2}\otimes(\boldsymbol{j} - \boldsymbol{\phi}) + (\boldsymbol{j} - \boldsymbol{\phi})\otimes\boldsymbol{I}_{m^2})\,\mathrm{d}\boldsymbol{\phi} \\
&\quad + (\boldsymbol{\psi}_0'\otimes\left(((\boldsymbol{I}_m - \boldsymbol{\Phi})\otimes(\boldsymbol{I}_m - \boldsymbol{\Phi}))\,(\boldsymbol{I}_{m^2} - \boldsymbol{\Phi}\otimes\boldsymbol{\Phi})^{-1}\right))(\boldsymbol{I}_m\otimes\mathrm{K}_m\otimes\boldsymbol{I}_m)(\boldsymbol{I}_{m^2}\otimes\boldsymbol{\phi} + \boldsymbol{\phi}\otimes\boldsymbol{I}_{m^2})\,\mathrm{d}\boldsymbol{\phi}
\end{aligned}$$

Recall the definition of $\boldsymbol{S}_1$to conclude that:

$$\mathrm{d}(\boldsymbol{S}_2)\,\mathrm{vec}\,\boldsymbol{\Sigma} = \boldsymbol{S}_1\,\mathrm{d}\boldsymbol{\phi}. \qquad \text{(Appendix A.3)}$$

Desired results follows by combining differential results for $\mathrm{dvec}\,\boldsymbol{\Theta}$ with proof of Lemma 3.4. □

Appendix A.1.4. Bimodality

Proof Theorem 3.2. Lets denote the true value for θ^2 as θ_0^2 that for general T is equal to:

$$\theta_0^2 = \sigma_0^2 + T(1-\phi_0)^2\,\mathrm{E}[u_{i,0}^2].$$

Thus at $T = 2$ it is equal to:

$$\theta_0^2 = \sigma_0^2 + 2(1-\phi_0)^2\,\mathrm{E}[u_{i,0}^2]$$

For some ϕ we denote the following variables:

$$\theta_\phi^2 = \mathrm{E}\left[\frac{2}{N}\sum_{i=1}^{N}(\ddot{y}_i - \phi\ddot{y}_{i-})^2\right], \quad \sigma_\phi^2 = \mathrm{E}\left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{2}(\tilde{y}_{i,t} - \phi\tilde{y}_{i,t-1})^2\right].$$

and $a = \phi_0 - \phi$.

As we assume that the observations are i.i.d. it is sufficient to analyze previous expressions for some arbitrary individual i. At first we proceed with expression for σ_ϕ^2 (recall definition of x variable):

$$\begin{aligned}
\sigma_\phi^2 &= \mathrm{E}\left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{2}(\tilde{y}_{i,t} - \phi\tilde{y}_{i,t-1})^2\right] = 0.5\,\mathrm{E}\left[(\Delta y_{i,2} - \phi\Delta y_{i,1})^2\right] = 0.5\,\mathrm{E}\left[(\Delta\varepsilon_{i,2} + (\phi_0-\phi)\Delta y_{i,1})^2\right] \\
&= 0.5\,\mathrm{E}\left[(\Delta\varepsilon_{i,2} + (\phi_0-\phi)((1-\phi_0)u_{i,0} + \varepsilon_{i,1}))^2\right] = 0.5\,\mathrm{E}\left[(\varepsilon_{i,2} + (\phi_0-\phi)(1-\phi_0)u_{i,0} + (\phi_0-\phi-1)\varepsilon_{i,1})^2\right] \\
&= 0.5(\sigma_0^2(1+(\phi_0-\phi-1)^2) + (\phi_0-\phi)^2(1-\phi_0)^2\,\mathrm{E}[u_{i,0}^2]) = 0.5\sigma_0^2\left(1 - 2(\phi_0-\phi) + 1 + (\phi_0-\phi)^2 x\right) \\
&= 0.5\sigma_0^2\left(a^2 x + 2(1-a)\right)
\end{aligned}$$

Similarly we can derive expression for θ_0^2 and θ_ϕ^2 in terms of the x and a.

$$\theta_0^2 = \sigma_0^2 + 2(1-\phi_0)^2\,\mathrm{E}[u_{i,0}^2] = \sigma_0^2\,(2x-1)$$

While for θ_ϕ^2 it follows that:

$$\begin{aligned}
\theta_\phi^2 &= \mathrm{E}\left[\frac{2}{N}\sum_{i=1}^{N}(\ddot{y}_i - \phi\ddot{y}_{i-})^2\right] = 2\,\mathrm{E}\left[(\bar{u}_i - u_{i,0} - \phi(\bar{u}_{i,-} - u_{i,0}))^2\right] \\
&= 2\,\mathrm{E}\left[(\bar{\varepsilon}_i + \phi_0\bar{u}_{i,-} - u_{i,0} - \phi(\bar{u}_{i,-} - u_{i,0}))^2\right] = 0.5\,\mathrm{E}\left[(\varepsilon_{i,2} + \varepsilon_{i,1} + \phi_0(u_{i,1} + u_{i,0}) - 2u_{i,0} - \phi(u_{i,1} - u_{i,0}))^2\right] \\
&= 0.5\,\mathrm{E}\left[(\varepsilon_{i,2} + \varepsilon_{i,1}(1+\phi_0-\phi) + u_{i,0}(\phi_0(1+\phi_0) - 2 - \phi(\phi_0-1)))^2\right] \\
&= 0.5\sigma_0^2\left[1 + (1+a)^2 + (1-\phi_0)^2\,\mathrm{E}[u_{i,0}^2](a+2)^2\right] = 0.5\sigma_0^2\left[1 + (1+a)^2 + (1-\phi_0)^2\,\mathrm{E}[u_{i,0}^2](a+2)^2/\sigma_0^2\right] \\
&= 0.5\sigma_0^2\left[1 + (1+a)^2 + (x-1)(a+2)^2\right] = 0.5\sigma_0^2\left[a^2 x + (a+1)(4x-2)\right]
\end{aligned}$$

Continuing:

$$\begin{aligned}
\sigma_\phi^2\theta_\phi^2 &= 0.25\sigma_0^4\left(a^2x - 2(a-1)\right)\left(a^2x + (a+1)(4x-2)\right) \\
&= 0.25\sigma_0^4\left(a^2\left(a^2x^2 + 2xa(2x-2) + (2x-2)^2\right) + 4(2x-1)\right) \\
&= 0.25\sigma_0^4\left(a^2\left(ax + 2(x-1)\right)^2 + 4(2x-1)\right) \\
&= 0.25\sigma_0^4\left(a^2\left(ax + 2(x-1)\right)^2\right) + \sigma_0^2\theta_0^2
\end{aligned}$$

The first term in the brackets is obviously equal for true value ϕ_0 ($a = 0$) and for:

$$a = 2\frac{1-x}{x} \Rightarrow \phi_0 - \phi = 2\frac{1-x}{x} \Rightarrow \phi = 2\frac{x-1}{x} + \phi_0.$$

$\square$

Appendix B. Monte Carlo results

Table B.1: Design 1

		$N=100$ $T=3$ $\pi=1$						$N=100$ $T=3$ $\pi=3$						$N=100$ $T=6$ $\pi=1$						$N=100$ $T=6$ $\pi=3$					
		Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE
ϕ_{11}	AB-GMM	-15.99	-15.57	-0.77	0.45	0.43	0.25	-36.00	-35.32	-1.29	0.53	0.69	0.43	-12.29	-11.78	-0.34	0.08	0.18	0.12	-19.76	-18.96	-0.48	0.06	0.26	0.19
	Sys-GMM	2.20	2.94	-0.24	0.25	0.15	0.10	17.04	18.22	-0.15	0.46	0.25	0.20	7.00	7.19	-0.06	0.19	0.10	0.08	25.34	26.17	0.11	0.37	0.27	0.26
	FDLS	0.33	0.21	-0.23	0.24	0.14	0.10	0.31	0.20	-0.23	0.24	0.14	0.10	0.11	0.05	-0.14	0.14	0.09	0.06	0.08	0.03	-0.14	0.14	0.09	0.06
	TMLE	6.00	3.23	-0.25	0.47	0.23	0.15	10.11	7.18	-0.25	0.53	0.26	0.16	1.78	0.81	-0.11	0.18	0.09	0.06	3.54	2.23	-0.11	0.22	0.11	0.06
	TMLEc	-0.51	-0.57	-0.20	0.19	0.12	0.08	-0.24	-0.48	-0.20	0.20	0.12	0.08	-0.34	-0.30	-0.10	0.09	0.06	0.04	-0.36	-0.31	-0.10	0.09	0.06	0.04
	TMLEs	1.41	-0.96	-0.26	0.37	0.19	0.13	1.46	-0.98	-0.26	0.37	0.19	0.13	0.88	0.19	-0.12	0.16	0.08	0.05	0.83	0.13	-0.12	0.16	0.08	0.05
	TMLEr	2.45	-0.44	-0.26	0.41	0.20	0.13	4.85	1.02	-0.26	0.48	0.23	0.14	0.88	0.19	-0.12	0.16	0.08	0.05	0.84	0.14	-0.12	0.16	0.08	0.05
ϕ_{12}	AB-GMM	-9.06	-8.52	-0.71	0.52	0.41	0.23	-19.61	-18.74	-1.18	0.75	0.66	0.37	-6.25	-6.11	-0.28	0.15	0.14	0.09	-11.35	-11.10	-0.40	0.16	0.20	0.14
	Sys-GMM	-1.02	-1.08	-0.25	0.23	0.15	0.09	-5.51	-5.63	-0.35	0.24	0.19	0.12	-2.29	-2.15	-0.14	0.09	0.08	0.05	-10.56	-10.97	-0.22	0.03	0.13	0.11
	FDLS	-0.14	-0.13	-0.25	0.24	0.15	0.10	-0.19	-0.18	-0.25	0.24	0.15	0.10	-0.15	-0.11	-0.16	0.16	0.10	0.06	-0.18	-0.14	-0.16	0.16	0.10	0.06
	TMLE	-2.77	-1.87	-0.31	0.22	0.16	0.11	-3.27	-2.02	-0.34	0.23	0.18	0.12	1.49	1.06	-0.10	0.14	0.08	0.05	3.00	2.64	-0.09	0.16	0.09	0.06
	TMLEc	-0.43	-0.32	-0.16	0.15	0.10	0.06	-0.62	-0.52	-0.17	0.15	0.10	0.06	-0.18	-0.21	-0.08	0.08	0.05	0.03	-0.19	-0.21	-0.08	0.08	0.05	0.03
	TMLEs	-3.88	-3.37	-0.28	0.19	0.15	0.10	-4.04	-3.54	-0.28	0.18	0.15	0.10	0.69	0.39	-0.10	0.12	0.07	0.05	0.64	0.33	-0.10	0.12	0.07	0.05
	TMLEr	-3.89	-3.43	-0.28	0.19	0.15	0.10	-4.77	-4.42	-0.29	0.18	0.15	0.10	0.69	0.39	-0.10	0.12	0.07	0.05	0.63	0.32	-0.10	0.12	0.07	0.05
		$N=250$ $T=3$ $\pi=1$						$N=250$ $T=3$ $\pi=3$						$N=250$ $T=6$ $\pi=1$						$N=250$ $T=6$ $\pi=3$					
ϕ_{11}	AB-GMM	-6.62	-7.01	-0.44	0.33	0.25	0.16	-21.20	-21.15	-0.92	0.48	0.50	0.30	-5.57	-5.39	-0.20	0.09	0.10	0.07	-11.00	-10.71	-0.33	0.09	0.17	0.12
	Sys-GMM	0.63	0.99	-0.16	0.16	0.10	0.07	10.01	10.71	-0.16	0.35	0.18	0.13	2.21	2.21	-0.06	0.10	0.06	0.04	15.61	15.74	0.01	0.30	0.18	0.16
	FDLS	0.04	-0.02	-0.15	0.15	0.09	0.06	0.03	-0.02	-0.15	0.15	0.09	0.06	0.05	0.16	-0.09	0.09	0.06	0.04	0.05	0.16	-0.09	0.09	0.06	0.04
	TMLE	3.60	1.05	-0.18	0.35	0.16	0.09	6.32	3.07	-0.17	0.42	0.19	0.10	1.08	0.30	-0.08	0.12	0.06	0.04	2.16	1.05	-0.07	0.16	0.07	0.04
	TMLEc	-0.33	-0.38	-0.13	0.12	0.07	0.05	-0.30	-0.36	-0.13	0.12	0.08	0.05	-0.16	-0.11	-0.06	0.06	0.04	0.02	-0.16	-0.11	-0.06	0.06	0.04	0.02
	TMLEs	1.43	-0.56	-0.18	0.28	0.14	0.09	1.47	-0.54	-0.18	0.28	0.14	0.09	0.76	0.14	-0.08	0.11	0.06	0.03	0.76	0.14	-0.08	0.11	0.06	0.03
	TMLEr	1.56	-0.51	-0.18	0.29	0.14	0.09	2.19	-0.31	-0.18	0.33	0.15	0.09	0.76	0.14	-0.08	0.11	0.06	0.03	0.76	0.14	-0.08	0.11	0.06	0.03
ϕ_{12}	AB-GMM	-3.78	-4.29	-0.41	0.35	0.24	0.15	-14.02	-14.11	-0.86	0.57	0.49	0.28	-2.93	-2.89	-0.17	0.11	0.09	0.06	-7.12	-6.90	-0.29	0.14	0.15	0.10
	Sys-GMM	-0.46	-0.35	-0.17	0.15	0.10	0.06	-2.00	-1.81	-0.28	0.22	0.15	0.09	-0.37	-0.31	-0.09	0.08	0.05	0.03	-4.61	-4.69	-0.18	0.09	0.09	0.07
	FDLS	0.02	0.01	-0.15	0.15	0.09	0.06	0.00	-0.01	-0.15	0.15	0.09	0.06	-0.04	0.00	-0.10	0.10	0.06	0.04	-0.04	0.00	-0.10	0.10	0.06	0.04
	TMLE	-0.28	0.25	-0.19	0.17	0.11	0.07	0.10	1.08	-0.23	0.19	0.13	0.08	1.02	0.33	-0.07	0.11	0.05	0.03	2.07	0.99	-0.06	0.13	0.06	0.04
	TMLEc	-0.15	-0.09	-0.10	0.10	0.06	0.04	-0.18	-0.12	-0.10	0.10	0.06	0.04	-0.10	-0.04	-0.05	0.05	0.03	0.02	-0.10	-0.04	-0.05	0.05	0.03	0.02
	TMLEs	-1.20	-0.92	-0.18	0.15	0.10	0.07	-1.27	-0.99	-0.18	0.15	0.10	0.07	0.71	0.20	-0.07	0.10	0.05	0.03	0.71	0.20	-0.07	0.10	0.05	0.03
	TMLEr	-1.21	-0.95	-0.18	0.15	0.10	0.07	-1.57	-1.25	-0.19	0.14	0.10	0.07	0.71	0.20	-0.07	0.10	0.05	0.03	0.71	0.20	-0.07	0.10	0.05	0.03

Table B.2: Design 2

		$N=100$ $T=3$ $\pi=1$						$N=100$ $T=3$ $\pi=3$						$N=100$ $T=6$ $\pi=1$						$N=100$ $T=6$ $\pi=3$					
		Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE
ϕ_{11}	AB-GMM	-8.01	-8.39	-0.54	0.42	0.31	0.20	-20.24	-20.72	-1.01	0.65	0.58	0.35	-7.19	-7.09	-0.29	0.14	0.15	0.10	-10.44	-10.38	-0.38	0.17	0.20	0.13
	Sys-GMM	3.98	4.07	-0.28	0.34	0.19	0.13	25.48	26.44	-0.19	0.68	0.37	0.28	10.15	9.97	-0.07	0.28	0.15	0.11	37.55	38.45	0.15	0.57	0.40	0.38
	FDLS	-4.67	-4.83	-0.34	0.25	0.18	0.13	-4.67	-4.82	-0.34	0.25	0.18	0.13	-5.43	-5.52	-0.24	0.14	0.13	0.09	-5.43	-5.53	-0.24	0.14	0.13	0.09
	TMLE	8.38	3.86	-0.27	0.59	0.27	0.16	14.88	9.62	-0.27	0.70	0.34	0.19	0.22	-0.13	-0.13	0.14	0.09	0.06	0.86	-0.06	-0.13	0.16	0.10	0.06
	TMLEc	2.25	2.17	-0.21	0.26	0.14	0.10	2.47	2.23	-0.21	0.26	0.15	0.10	0.58	0.56	-0.11	0.12	0.07	0.05	0.58	0.56	-0.11	0.12	0.07	0.05
	TMLEs	5.35	1.67	-0.27	0.51	0.24	0.14	6.41	2.36	-0.27	0.55	0.25	0.14	0.18	-0.13	-0.13	0.14	0.08	0.06	0.18	-0.13	-0.13	0.14	0.08	0.06
	TMLEr	5.61	1.75	-0.27	0.52	0.24	0.14	8.11	3.14	-0.27	0.60	0.27	0.15	0.18	-0.13	-0.13	0.14	0.08	0.06	0.19	-0.13	-0.13	0.14	0.08	0.06
ϕ_{12}	AB-GMM	-1.74	-1.41	-0.50	0.45	0.29	0.18	0.00	0.01	-0.86	0.88	0.57	0.32	-0.43	-0.34	-0.22	0.20	0.13	0.09	0.42	0.48	-0.28	0.28	0.17	0.12
	Sys-GMM	-1.67	-1.64	-0.33	0.29	0.19	0.13	-8.19	-8.47	-0.51	0.35	0.27	0.18	-2.94	-2.86	-0.20	0.14	0.11	0.07	-9.99	-10.00	-0.29	0.10	0.15	0.11
	FDLS	4.98	4.98	-0.27	0.36	0.20	0.13	4.97	4.97	-0.27	0.36	0.20	0.13	8.36	8.39	-0.12	0.29	0.15	0.10	8.37	8.39	-0.12	0.29	0.15	0.10
	TMLE	-0.88	-0.55	-0.38	0.34	0.21	0.13	-2.86	-1.52	-0.46	0.37	0.25	0.16	0.05	0.10	-0.12	0.12	0.07	0.05	0.03	0.17	-0.13	0.13	0.09	0.05
	TMLEc	-7.27	-7.32	-0.27	0.13	0.14	0.10	-7.27	-7.31	-0.27	0.13	0.14	0.10	-3.31	-3.35	-0.14	0.07	0.07	0.05	-3.31	-3.35	-0.14	0.07	0.07	0.05
	TMLEs	-0.69	-0.55	-0.32	0.30	0.19	0.12	-1.35	-1.15	-0.32	0.29	0.18	0.12	0.01	0.09	-0.12	0.12	0.07	0.05	0.00	0.09	-0.12	0.12	0.07	0.05
	TMLEr	-0.66	-0.53	-0.32	0.30	0.18	0.12	-1.91	-1.71	-0.32	0.28	0.18	0.12	0.01	0.09	-0.12	0.12	0.07	0.05	0.00	0.09	-0.12	0.12	0.07	0.05
		$N=250$ $T=3$ $\pi=1$						$N=250$ $T=3$ $\pi=3$						$N=250$ $T=6$ $\pi=1$						$N=250$ $T=6$ $\pi=3$					
ϕ_{11}	AB-GMM	-3.39	-4.05	-0.33	0.28	0.19	0.13	-8.81	-10.09	-0.62	0.49	0.35	0.22	-3.16	-3.21	-0.17	0.11	0.09	0.06	-4.99	-4.97	-0.24	0.14	0.13	0.08
	Sys-GMM	1.37	1.41	-0.19	0.21	0.12	0.08	14.37	14.13	-0.19	0.48	0.25	0.17	2.91	2.86	-0.08	0.14	0.07	0.05	22.59	22.34	0.02	0.44	0.26	0.22
	FDLS	-5.11	-5.13	-0.23	0.13	0.12	0.08	-5.11	-5.13	-0.23	0.13	0.12	0.08	-5.62	-5.73	-0.17	0.06	0.09	0.07	-5.62	-5.73	-0.17	0.06	0.09	0.07
	TMLE	4.06	0.78	-0.18	0.42	0.18	0.09	8.10	2.55	-0.18	0.57	0.24	0.11	0.02	-0.09	-0.08	0.08	0.05	0.03	0.03	-0.09	-0.08	0.08	0.05	0.03
	TMLEc	2.30	2.29	-0.12	0.17	0.09	0.06	2.32	2.30	-0.12	0.17	0.09	0.06	0.70	0.68	-0.07	0.08	0.04	0.03	0.70	0.68	-0.07	0.08	0.04	0.03
	TMLEs	2.87	0.40	-0.18	0.35	0.16	0.09	3.17	0.50	-0.18	0.37	0.17	0.09	0.02	-0.09	-0.08	0.08	0.05	0.03	0.02	-0.09	-0.08	0.08	0.05	0.03
	TMLEr	2.91	0.40	-0.18	0.35	0.16	0.09	3.55	0.56	-0.18	0.41	0.18	0.09	0.02	-0.09	-0.08	0.08	0.05	0.03	0.02	-0.09	-0.08	0.08	0.05	0.03
ϕ_{12}	AB-GMM	-0.66	-0.80	-0.30	0.29	0.18	0.12	0.39	-0.13	-0.56	0.56	0.35	0.21	-0.06	-0.05	-0.14	0.14	0.09	0.06	0.59	0.75	-0.19	0.20	0.12	0.08
	Sys-GMM	-0.59	-0.62	-0.21	0.19	0.12	0.08	-5.03	-4.67	-0.39	0.28	0.21	0.14	-0.74	-0.75	-0.12	0.10	0.07	0.04	-5.98	-5.95	-0.26	0.14	0.13	0.09
	FDLS	5.09	5.14	-0.14	0.25	0.13	0.09	5.09	5.14	-0.14	0.25	0.13	0.09	8.49	8.52	-0.05	0.21	0.12	0.09	8.49	8.52	-0.05	0.21	0.12	0.09
	TMLE	0.85	0.32	-0.22	0.26	0.14	0.08	-0.27	0.19	-0.35	0.30	0.18	0.10	0.00	0.04	-0.08	0.07	0.05	0.03	0.01	0.04	-0.08	0.07	0.05	0.03
	TMLEc	-7.28	-7.26	-0.20	0.05	0.11	0.08	-7.28	-7.27	-0.20	0.05	0.11	0.08	-3.29	-3.24	-0.10	0.03	0.05	0.04	-3.29	-3.24	-0.10	0.03	0.05	0.04
	TMLEs	0.67	0.13	-0.19	0.23	0.13	0.08	0.47	-0.04	-0.19	0.23	0.13	0.08	0.00	0.04	-0.08	0.07	0.05	0.03	0.00	0.04	-0.08	0.07	0.05	0.03
	TMLEr	0.66	0.12	-0.19	0.23	0.13	0.08	0.24	-0.19	-0.20	0.22	0.13	0.08	0.00	0.04	-0.08	0.07	0.05	0.03	0.00	0.04	-0.08	0.07	0.05	0.03

Table B.3: Design 3

		$N=100$ $T=3$ $\pi=1$						$N=100$ $T=3$ $\pi=3$						$N=100$ $T=6$ $\pi=1$						$N=100$ $T=6$ $\pi=3$					
		Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE
ϕ_{11}	AB-GMM	-27.55	-22.63	-1.33	0.64	0.71	0.36	-3.27	-2.29	-0.33	0.23	0.20	0.09	-11.65	-11.26	-0.41	0.16	0.21	0.14	-5.37	-4.34	-0.26	0.12	0.13	0.07
	Sys-GMM	29.51	30.90	0.03	0.51	0.33	0.31	60.94	61.50	0.51	0.70	0.61	0.62	29.03	29.54	0.11	0.46	0.31	0.30	57.84	58.32	0.51	0.63	0.58	0.58
	FDLS	10.03	9.89	-0.21	0.42	0.22	0.14	64.34	64.56	0.27	1.01	0.68	0.65	0.90	0.77	-0.18	0.21	0.12	0.08	34.68	33.57	0.06	0.66	0.39	0.34
	TMLE	16.37	7.55	-0.24	0.81	0.36	0.17	7.47	0.73	-0.13	0.78	0.27	0.06	0.32	-0.09	-0.12	0.13	0.09	0.05	0.07	-0.14	-0.08	0.08	0.06	0.03
	TMLEc	17.70	17.43	-0.09	0.46	0.24	0.18	52.13	51.55	0.34	0.79	0.54	0.52	7.81	7.54	-0.06	0.22	0.12	0.08	40.94	42.57	0.19	0.57	0.43	0.43
	TMLEs	4.00	0.72	-0.23	0.45	0.21	0.11	0.20	-0.21	-0.13	0.14	0.09	0.05	0.01	-0.11	-0.12	0.13	0.08	0.05	-0.14	-0.15	-0.08	0.08	0.05	0.03
	TMLEr	5.27	0.94	-0.23	0.55	0.23	0.11	0.45	-0.21	-0.13	0.15	0.10	0.05	0.01	-0.11	-0.12	0.13	0.08	0.05	-0.14	-0.15	-0.08	0.08	0.05	0.03
ϕ_{12}	AB-GMM	-1.70	-3.29	-1.03	1.03	0.67	0.33	0.61	0.39	-0.24	0.25	0.17	0.08	-0.74	-0.65	-0.30	0.28	0.18	0.11	1.18	1.05	-0.17	0.19	0.11	0.06
	Sys-GMM	-3.10	-3.76	-0.23	0.20	0.13	0.09	-10.43	-10.60	-0.17	-0.03	0.11	0.11	-4.45	-4.66	-0.18	0.10	0.09	0.07	-11.71	-11.88	-0.16	-0.07	0.12	0.12
	FDLS	5.65	5.50	-0.26	0.38	0.20	0.13	9.04	9.12	-0.22	0.40	0.21	0.14	8.55	8.58	-0.12	0.29	0.15	0.11	9.82	9.80	-0.15	0.35	0.18	0.12
	TMLE	-0.76	-0.06	-0.49	0.43	0.26	0.15	0.83	0.22	-0.17	0.30	0.17	0.05	0.06	0.09	-0.11	0.11	0.07	0.04	0.08	0.07	-0.08	0.07	0.05	0.03
	TMLEc	-7.42	-7.50	-0.29	0.15	0.15	0.10	-3.83	-4.86	-0.16	0.12	0.09	0.07	-3.07	-3.03	-0.15	0.09	0.08	0.05	-5.02	-5.96	-0.18	0.11	0.10	0.08
	TMLEs	-0.07	-0.33	-0.25	0.26	0.16	0.09	-0.04	-0.04	-0.12	0.12	0.07	0.05	-0.05	0.06	-0.11	0.11	0.07	0.04	-0.01	0.06	-0.08	0.07	0.05	0.03
	TMLEr	-0.36	-0.60	-0.25	0.25	0.15	0.09	-0.07	-0.05	-0.12	0.12	0.07	0.05	-0.05	0.06	-0.11	0.11	0.07	0.04	-0.01	0.06	-0.08	0.07	0.05	0.03

		$N=250$ $T=3$ $\pi=1$						$N=250$ $T=3$ $\pi=3$						$N=250$ $T=6$ $\pi=1$						$N=250$ $T=6$ $\pi=3$					
ϕ_{11}	AB-GMM	-17.44	-13.68	-0.99	0.51	0.54	0.26	-0.78	-0.78	-0.14	0.13	0.08	0.05	-5.85	-5.68	-0.26	0.13	0.13	0.09	-1.95	-1.63	-0.13	0.08	0.07	0.04
	Sys-GMM	32.06	32.74	0.16	0.46	0.33	0.33	61.17	61.34	0.55	0.67	0.61	0.61	27.44	27.71	0.14	0.41	0.29	0.28	57.84	58.01	0.54	0.61	0.58	0.58
	FDLS	9.87	9.79	-0.10	0.30	0.16	0.11	66.92	67.01	0.42	0.92	0.69	0.67	0.75	0.60	-0.11	0.13	0.07	0.05	35.92	35.26	0.17	0.57	0.38	0.35
	TMLE	9.61	2.14	-0.15	0.69	0.26	0.09	1.10	0.01	-0.08	0.09	0.11	0.03	-0.03	-0.07	-0.07	0.08	0.05	0.03	-0.06	-0.11	-0.05	0.05	0.03	0.02
	TMLEc	18.19	17.87	0.01	0.36	0.21	0.18	51.37	50.86	0.42	0.60	0.52	0.51	7.98	7.85	-0.01	0.17	0.10	0.08	44.00	44.97	0.30	0.55	0.45	0.45
	TMLEs	1.26	-0.06	-0.15	0.21	0.12	0.07	-0.02	-0.07	-0.08	0.08	0.05	0.03	-0.03	-0.07	-0.07	0.08	0.05	0.03	-0.06	-0.11	-0.05	0.05	0.03	0.02
	TMLEr	1.62	-0.02	-0.15	0.22	0.13	0.07	-0.02	-0.07	-0.08	0.08	0.05	0.03	-0.03	-0.07	-0.07	0.08	0.05	0.03	-0.06	-0.11	-0.05	0.05	0.03	0.02
ϕ_{12}	AB-GMM	0.12	-1.50	-0.75	0.82	0.52	0.25	0.10	0.11	-0.12	0.12	0.08	0.05	-0.25	-0.13	-0.21	0.20	0.13	0.08	0.49	0.50	-0.10	0.11	0.06	0.04
	Sys-GMM	-3.98	-4.26	-0.16	0.09	0.09	0.06	-10.44	-10.49	-0.15	-0.06	0.11	0.10	-3.81	-3.98	-0.14	0.07	0.07	0.05	-11.78	-11.83	-0.14	-0.09	0.12	0.12
	FDLS	5.88	5.82	-0.14	0.26	0.13	0.09	9.45	9.47	-0.11	0.29	0.15	0.11	8.71	8.76	-0.04	0.22	0.12	0.09	10.09	10.22	-0.06	0.26	0.14	0.11
	TMLE	0.60	0.41	-0.38	0.37	0.19	0.08	0.13	-0.01	-0.07	0.07	0.06	0.03	-0.01	-0.02	-0.07	0.07	0.04	0.03	-0.01	-0.01	-0.05	0.05	0.03	0.02
	TMLEc	-7.59	-7.53	-0.22	0.06	0.11	0.08	-4.78	-5.24	-0.11	0.04	0.07	0.06	-3.01	-2.96	-0.11	0.04	0.06	0.04	-6.36	-6.87	-0.15	0.04	0.09	0.07
	TMLEs	0.32	-0.05	-0.14	0.15	0.09	0.06	-0.02	-0.04	-0.07	0.07	0.04	0.03	-0.01	-0.02	-0.07	0.07	0.04	0.03	-0.01	-0.01	-0.05	0.05	0.03	0.02
	TMLEr	0.19	-0.13	-0.14	0.15	0.09	0.06	-0.02	-0.04	-0.07	0.07	0.04	0.03	-0.01	-0.02	-0.07	0.07	0.04	0.03	-0.01	-0.01	-0.05	0.05	0.03	0.02

Table B.4: Design 4

		$N=100$ $T=3$ $\pi=1$						$N=100$ $T=3$ $\pi=3$						$N=100$ $T=6$ $\pi=1$						$N=100$ $T=6$ $\pi=3$					
		Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE
ϕ_{11}	AB-GMM	-2.40	-2.64	-0.33	0.30	0.19	0.13	-0.60	-0.67	-0.19	0.18	0.11	0.07	-3.95	-4.05	-0.21	0.12	0.11	0.07	-1.59	-1.58	-0.13	0.09	0.07	0.04
	Sys-GMM	28.82	29.78	-0.11	0.66	0.37	0.30	52.08	51.33	0.46	0.61	0.52	0.51	29.15	30.16	0.03	0.52	0.33	0.30	52.80	52.70	0.50	0.56	0.53	0.53
	FDLS	7.10	6.79	-0.23	0.39	0.20	0.13	56.24	56.13	0.17	0.96	0.61	0.56	0.33	0.22	-0.19	0.20	0.12	0.08	32.14	31.26	0.04	0.63	0.37	0.31
	TMLE	12.18	4.85	-0.25	0.73	0.32	0.15	11.09	1.69	-0.18	0.96	0.33	0.08	0.62	-0.12	-0.12	0.14	0.10	0.05	0.32	-0.01	-0.09	0.09	0.07	0.04
	TMLEc	13.52	13.23	-0.12	0.40	0.21	0.15	48.70	47.53	0.21	0.84	0.52	0.48	6.98	6.83	-0.07	0.21	0.11	0.08	36.26	36.95	0.15	0.55	0.38	0.37
	TMLEs	4.88	1.30	-0.24	0.51	0.22	0.12	0.80	0.14	-0.15	0.18	0.11	0.06	0.11	-0.14	-0.12	0.13	0.08	0.05	0.02	-0.02	-0.09	0.09	0.05	0.04
	TMLEr	5.42	1.44	-0.24	0.54	0.23	0.12	1.48	0.17	-0.15	0.19	0.14	0.06	0.11	-0.14	-0.12	0.13	0.08	0.05	0.02	-0.02	-0.09	0.09	0.05	0.04
ϕ_{12}	AB-GMM	-0.46	-0.27	-0.32	0.30	0.19	0.13	-0.14	0.01	-0.20	0.20	0.12	0.08	-0.25	-0.24	-0.17	0.16	0.10	0.07	-0.06	-0.09	-0.11	0.11	0.07	0.04
	Sys-GMM	-13.25	-13.52	-0.43	0.17	0.23	0.16	-16.05	-15.48	-0.25	-0.09	0.17	0.15	-9.47	-9.78	-0.28	0.10	0.15	0.11	-14.87	-14.80	-0.18	-0.12	0.15	0.15
	FDLS	5.69	5.80	-0.26	0.38	0.20	0.14	8.76	8.69	-0.25	0.43	0.23	0.15	8.56	8.69	-0.12	0.29	0.15	0.11	9.64	9.67	-0.15	0.34	0.18	0.12
	TMLE	-0.26	0.07	-0.44	0.40	0.24	0.14	1.07	0.40	-0.50	0.52	0.26	0.09	0.14	0.04	-0.12	0.12	0.08	0.05	0.13	-0.05	-0.09	0.09	0.06	0.03
	TMLEc	-6.02	-6.03	-0.28	0.16	0.14	0.10	0.37	-0.08	-0.20	0.23	0.13	0.08	-2.74	-2.68	-0.14	0.09	0.08	0.05	-0.54	-1.08	-0.17	0.18	0.11	0.07
	TMLEs	-0.18	-0.31	-0.28	0.29	0.17	0.11	-0.08	0.01	-0.16	0.16	0.10	0.06	-0.03	-0.05	-0.11	0.11	0.07	0.05	-0.07	-0.06	-0.09	0.08	0.05	0.03
	TMLEr	-0.19	-0.36	-0.28	0.29	0.17	0.11	-0.13	-0.02	-0.16	0.16	0.10	0.06	-0.03	-0.05	-0.11	0.11	0.07	0.05	-0.07	-0.06	-0.09	0.08	0.05	0.03
		$N=250$ $T=3$ $\pi=1$						$N=250$ $T=3$ $\pi=3$						$N=250$ $T=6$ $\pi=1$						$N=250$ $T=6$ $\pi=3$					
ϕ_{11}	AB-GMM	-0.98	-1.20	-0.20	0.19	0.12	0.08	-0.22	-0.35	-0.11	0.11	0.07	0.04	-1.62	-1.66	-0.12	0.09	0.07	0.04	-0.57	-0.53	-0.07	0.06	0.04	0.03
	Sys-GMM	35.13	36.30	0.01	0.66	0.40	0.36	52.40	52.05	0.48	0.58	0.52	0.52	31.43	32.22	0.08	0.53	0.34	0.32	53.91	53.79	0.52	0.56	0.54	0.54
	FDLS	6.86	6.73	-0.12	0.26	0.14	0.09	58.32	58.17	0.33	0.84	0.60	0.58	0.19	0.03	-0.12	0.13	0.07	0.05	33.21	32.54	0.15	0.54	0.35	0.33
	TMLE	9.10	2.08	-0.16	0.69	0.26	0.09	2.75	0.10	-0.11	0.17	0.17	0.04	0.05	-0.09	-0.07	0.08	0.05	0.03	0.01	-0.03	-0.05	0.05	0.03	0.02
	TMLEc	13.89	13.79	-0.02	0.30	0.17	0.14	50.21	47.74	0.32	0.79	0.52	0.48	7.16	7.13	-0.01	0.16	0.09	0.07	38.81	39.19	0.25	0.52	0.40	0.39
	TMLEs	1.88	0.30	-0.16	0.24	0.14	0.07	0.12	-0.01	-0.10	0.10	0.06	0.04	0.01	-0.09	-0.07	0.08	0.05	0.03	0.00	-0.03	-0.05	0.05	0.03	0.02
	TMLEr	2.04	0.35	-0.16	0.25	0.14	0.07	0.16	-0.01	-0.10	0.10	0.06	0.04	0.01	-0.09	-0.07	0.08	0.05	0.03	0.00	-0.03	-0.05	0.05	0.03	0.02
ϕ_{12}	AB-GMM	-0.11	-0.13	-0.20	0.19	0.12	0.08	0.01	-0.10	-0.12	0.12	0.07	0.05	-0.02	-0.04	-0.10	0.11	0.06	0.04	0.01	-0.01	-0.07	0.07	0.04	0.03
	Sys-GMM	-16.32	-16.26	-0.39	0.06	0.21	0.17	-16.06	-15.83	-0.21	-0.12	0.16	0.16	-10.31	-10.30	-0.26	0.06	0.14	0.11	-15.31	-15.25	-0.17	-0.13	0.15	0.15
	FDLS	5.82	5.71	-0.14	0.26	0.13	0.09	9.05	9.01	-0.13	0.31	0.16	0.11	8.68	8.70	-0.05	0.22	0.12	0.09	9.82	9.86	-0.06	0.26	0.14	0.10
	TMLE	0.95	0.74	-0.38	0.37	0.20	0.09	1.13	0.23	-0.12	0.21	0.15	0.04	0.03	0.06	-0.07	0.07	0.04	0.03	0.00	0.02	-0.05	0.05	0.03	0.02
	TMLEc	-6.04	-6.08	-0.20	0.08	0.10	0.07	-0.24	-0.46	-0.13	0.13	0.08	0.05	-2.71	-2.71	-0.10	0.05	0.05	0.04	-1.00	-1.28	-0.12	0.11	0.07	0.05
	TMLEs	0.63	0.12	-0.16	0.19	0.11	0.06	0.05	0.02	-0.10	0.10	0.06	0.04	0.00	0.05	-0.07	0.07	0.04	0.03	-0.01	0.02	-0.05	0.05	0.03	0.02
	TMLEr	0.58	0.10	-0.16	0.19	0.11	0.06	0.05	0.02	-0.10	0.10	0.06	0.04	0.00	0.05	-0.07	0.07	0.04	0.03	-0.01	0.02	-0.05	0.05	0.03	0.02

Table B.5: Design 5

		$N=100$ $T=3$ $\pi=1$						$N=100$ $T=3$ $\pi=3$						$N=100$ $T=6$ $\pi=1$						$N=100$ $T=6$ $\pi=3$					
		Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE
ϕ_{11}	AB-GMM	-14.65	-15.65	-0.77	0.51	0.43	0.27	-29.43	-30.04	-1.31	0.75	0.74	0.45	-12.94	-13.00	-0.40	0.14	0.21	0.15	-15.89	-15.89	-0.49	0.17	0.26	0.18
	Sys-GMM	6.86	7.10	-0.33	0.45	0.25	0.17	32.33	34.14	-0.19	0.77	0.44	0.35	18.43	18.43	-0.05	0.41	0.23	0.19	46.43	47.69	0.26	0.62	0.48	0.48
	FDLS	-4.67	-4.75	-0.46	0.37	0.26	0.17	-4.69	-4.76	-0.46	0.37	0.26	0.17	-5.52	-5.59	-0.32	0.21	0.17	0.11	-5.53	-5.61	-0.32	0.21	0.17	0.11
	TMLE	10.61	6.59	-0.37	0.69	0.34	0.22	17.63	14.98	-0.36	0.78	0.40	0.27	1.02	-0.23	-0.19	0.24	0.14	0.08	3.62	0.38	-0.19	0.44	0.19	0.09
	TMLEc	1.69	1.25	-0.30	0.35	0.20	0.13	2.38	1.62	-0.30	0.38	0.21	0.13	0.23	0.35	-0.16	0.17	0.10	0.07	0.24	0.33	-0.16	0.17	0.10	0.07
	TMLEs	5.13	1.49	-0.36	0.58	0.29	0.19	6.97	3.06	-0.35	0.61	0.30	0.19	0.39	-0.34	-0.18	0.21	0.13	0.08	0.56	-0.30	-0.18	0.22	0.13	0.08
	TMLEr	5.88	1.80	-0.36	0.60	0.30	0.19	10.70	5.46	-0.35	0.70	0.34	0.22	0.41	-0.34	-0.18	0.21	0.13	0.08	0.86	-0.27	-0.18	0.23	0.14	0.08
ϕ_{12}	AB-GMM	-2.69	-2.54	-0.67	0.60	0.42	0.24	-0.96	0.06	-1.10	1.07	0.72	0.38	0.16	0.40	-0.28	0.28	0.17	0.11	1.25	1.36	-0.33	0.36	0.21	0.14
	Sys-GMM	-2.75	-2.70	-0.42	0.36	0.24	0.16	-9.82	-9.87	-0.56	0.38	0.30	0.19	-5.09	-5.07	-0.26	0.16	0.14	0.09	-12.01	-12.29	-0.28	0.05	0.16	0.13
	FDLS	5.41	5.54	-0.40	0.50	0.28	0.18	5.41	5.54	-0.40	0.50	0.28	0.18	8.65	8.72	-0.22	0.38	0.20	0.14	8.64	8.71	-0.22	0.38	0.20	0.14
	TMLE	-1.70	-0.89	-0.48	0.42	0.27	0.18	-3.84	-2.94	-0.53	0.43	0.29	0.20	0.38	0.20	-0.18	0.20	0.12	0.07	-0.26	0.21	-0.27	0.23	0.15	0.08
	TMLEc	-6.94	-6.86	-0.35	0.21	0.18	0.12	-6.92	-6.81	-0.35	0.20	0.18	0.12	-3.21	-3.20	-0.18	0.12	0.10	0.07	-3.22	-3.21	-0.18	0.12	0.10	0.07
	TMLEs	-1.33	-1.06	-0.40	0.36	0.23	0.15	-2.43	-2.22	-0.39	0.34	0.22	0.15	0.28	0.14	-0.17	0.18	0.11	0.07	0.12	0.04	-0.17	0.18	0.11	0.07
	TMLEr	-1.27	-0.97	-0.40	0.37	0.23	0.15	-2.97	-2.99	-0.39	0.34	0.22	0.15	0.27	0.13	-0.17	0.18	0.11	0.07	-0.04	-0.09	-0.18	0.18	0.11	0.07
		$N=250$ $T=3$ $\pi=1$						$N=250$ $T=3$ $\pi=3$						$N=250$ $T=6$ $\pi=1$						$N=250$ $T=6$ $\pi=3$					
ϕ_{11}	AB-GMM	-6.59	-7.18	-0.47	0.37	0.27	0.18	-15.32	-16.18	-0.88	0.62	0.50	0.31	-5.79	-5.61	-0.25	0.13	0.13	0.09	-7.80	-7.71	-0.32	0.16	0.16	0.11
	Sys-GMM	2.78	2.74	-0.24	0.29	0.16	0.11	22.13	22.68	-0.19	0.64	0.34	0.25	8.51	8.18	-0.07	0.25	0.13	0.09	37.35	38.00	0.15	0.57	0.40	0.38
	FDLS	-5.19	-5.06	-0.30	0.20	0.16	0.11	-5.19	-5.07	-0.30	0.20	0.16	0.11	-5.67	-5.67	-0.22	0.11	0.11	0.08	-5.67	-5.67	-0.22	0.11	0.11	0.08
	TMLE	6.35	1.63	-0.24	0.54	0.24	0.13	12.30	5.86	-0.24	0.67	0.31	0.17	-0.03	-0.23	-0.12	0.12	0.07	0.05	0.25	-0.21	-0.12	0.13	0.08	0.05
	TMLEc	2.01	1.84	-0.18	0.23	0.12	0.08	2.08	1.86	-0.18	0.23	0.13	0.08	0.52	0.59	-0.10	0.11	0.06	0.04	0.52	0.59	-0.10	0.11	0.06	0.04
	TMLEs	4.03	0.40	-0.24	0.47	0.21	0.12	4.91	0.84	-0.24	0.51	0.22	0.13	-0.05	-0.23	-0.12	0.12	0.07	0.05	-0.04	-0.23	-0.12	0.12	0.07	0.05
	TMLEr	4.16	0.43	-0.24	0.48	0.22	0.12	6.03	1.14	-0.24	0.55	0.24	0.13	-0.05	-0.23	-0.12	0.12	0.07	0.05	-0.04	-0.23	-0.12	0.12	0.07	0.05
ϕ_{12}	AB-GMM	-0.76	-0.86	-0.41	0.40	0.25	0.16	0.76	0.33	-0.75	0.79	0.49	0.29	0.13	0.16	-0.19	0.19	0.11	0.08	1.04	1.05	-0.24	0.26	0.15	0.10
	Sys-GMM	-1.12	-1.07	-0.28	0.26	0.16	0.11	-7.53	-7.62	-0.48	0.33	0.26	0.17	-2.29	-2.21	-0.17	0.13	0.10	0.06	-9.96	-10.34	-0.29	0.11	0.16	0.12
	FDLS	5.34	5.28	-0.22	0.33	0.17	0.12	5.34	5.28	-0.22	0.33	0.17	0.12	8.60	8.66	-0.10	0.27	0.14	0.10	8.60	8.66	-0.10	0.27	0.14	0.10
	TMLE	0.22	0.41	-0.33	0.32	0.19	0.12	-1.69	-0.35	-0.44	0.35	0.23	0.14	0.13	0.13	-0.10	0.11	0.07	0.04	0.17	0.15	-0.11	0.11	0.07	0.04
	TMLEc	-7.10	-7.10	-0.25	0.10	0.13	0.09	-7.10	-7.09	-0.25	0.10	0.13	0.09	-3.23	-3.21	-0.13	0.06	0.07	0.04	-3.23	-3.21	-0.13	0.06	0.07	0.04
	TMLEs	0.30	0.14	-0.27	0.29	0.17	0.10	-0.30	-0.24	-0.27	0.28	0.17	0.10	0.11	0.13	-0.10	0.11	0.06	0.04	0.11	0.13	-0.10	0.11	0.06	0.04
	TMLEr	0.31	0.15	-0.27	0.29	0.17	0.10	-0.74	-0.67	-0.28	0.27	0.17	0.10	0.12	0.13	-0.10	0.11	0.06	0.04	0.11	0.13	-0.10	0.11	0.06	0.04

Table B.6: Design 6

		$N=100$ $T=3$ $\pi=1$						$N=100$ $T=3$ $\pi=3$						$N=100$ $T=6$ $\pi=1$						$N=100$ $T=6$ $\pi=3$					
		Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE	Mean	Median	5 q	95 q	RMSE	MAE
ϕ_{11}	AB-GMM	-8.23	-8.40	-0.55	0.41	0.31	0.19	-25.65	-25.21	-1.19	0.67	0.66	0.39	-8.24	-8.10	-0.31	0.14	0.16	0.11	-12.27	-12.27	-0.43	0.17	0.22	0.15
	Sys-GMM	6.44	6.27	-0.26	0.39	0.21	0.14	30.62	31.82	-0.15	0.73	0.41	0.33	10.50	10.36	-0.07	0.28	0.15	0.11	37.92	38.86	0.16	0.57	0.40	0.39
	FDLS	0.30	0.13	-0.31	0.31	0.19	0.13	0.27	0.12	-0.31	0.31	0.19	0.13	-6.69	-6.74	-0.26	0.13	0.14	0.10	-6.69	-6.74	-0.26	0.13	0.14	0.10
	TMLE	12.75	9.93	-0.21	0.56	0.27	0.16	17.42	13.80	-0.21	0.67	0.32	0.19	3.01	2.77	-0.10	0.18	0.09	0.06	3.12	2.78	-0.10	0.18	0.09	0.06
	TMLEc	10.65	10.63	-0.13	0.35	0.18	0.13	10.91	10.70	-0.13	0.36	0.19	0.13	2.98	2.90	-0.09	0.15	0.08	0.05	2.98	2.90	-0.09	0.15	0.08	0.05
	TMLEs	10.66	8.50	-0.21	0.50	0.24	0.14	11.75	9.22	-0.21	0.53	0.25	0.15	3.00	2.77	-0.10	0.18	0.09	0.06	3.00	2.77	-0.10	0.18	0.09	0.06
	TMLEr	10.88	8.58	-0.21	0.51	0.24	0.15	13.12	9.76	-0.21	0.58	0.27	0.15	3.00	2.77	-0.10	0.18	0.09	0.06	3.00	2.77	-0.10	0.18	0.09	0.06
ϕ_{12}	AB-GMM	-2.01	-1.59	-0.49	0.43	0.29	0.18	-0.04	-0.51	-0.96	0.97	0.63	0.33	-0.62	-0.48	-0.23	0.21	0.14	0.09	0.25	0.28	-0.31	0.31	0.19	0.12
	Sys-GMM	-2.17	-2.07	-0.35	0.29	0.20	0.13	-9.12	-9.49	-0.52	0.35	0.28	0.18	-2.87	-2.83	-0.20	0.14	0.11	0.07	-9.84	-9.90	-0.29	0.10	0.15	0.11
	FDLS	4.04	4.02	-0.29	0.36	0.20	0.14	4.05	4.05	-0.29	0.36	0.20	0.14	8.24	8.38	-0.13	0.30	0.15	0.11	8.25	8.38	-0.13	0.30	0.15	0.11
	TMLE	-0.99	-0.31	-0.34	0.30	0.19	0.12	-2.87	-1.67	-0.42	0.31	0.22	0.14	-0.18	-0.14	-0.13	0.12	0.07	0.05	-0.19	-0.13	-0.13	0.12	0.08	0.05
	TMLEc	-8.75	-8.81	-0.29	0.12	0.15	0.11	-8.70	-8.71	-0.29	0.12	0.15	0.11	-3.77	-3.75	-0.14	0.07	0.08	0.05	-3.77	-3.75	-0.14	0.07	0.08	0.05
	TMLEs	-0.51	-0.16	-0.30	0.28	0.17	0.11	-1.23	-0.74	-0.30	0.26	0.17	0.11	-0.18	-0.14	-0.13	0.12	0.07	0.05	-0.18	-0.14	-0.13	0.12	0.07	0.05
	TMLEr	-0.56	-0.19	-0.30	0.28	0.17	0.11	-1.89	-1.58	-0.31	0.26	0.17	0.11	-0.18	-0.14	-0.13	0.12	0.07	0.05	-0.19	-0.14	-0.13	0.12	0.07	0.05
		$N=250$ $T=3$ $\pi=1$						$N=250$ $T=3$ $\pi=3$						$N=250$ $T=6$ $\pi=1$						$N=250$ $T=6$ $\pi=3$					
ϕ_{11}	AB-GMM	-3.43	-3.83	-0.33	0.27	0.19	0.12	-12.02	-12.61	-0.73	0.51	0.41	0.25	-3.54	-3.45	-0.18	0.11	0.10	0.07	-5.90	-5.90	-0.27	0.15	0.14	0.09
	Sys-GMM	2.64	2.44	-0.18	0.24	0.13	0.09	19.39	18.87	-0.16	0.57	0.30	0.21	3.04	3.03	-0.07	0.14	0.07	0.05	22.62	22.26	0.02	0.44	0.26	0.22
	FDLS	-0.12	-0.19	-0.19	0.19	0.12	0.08	-0.12	-0.18	-0.19	0.19	0.12	0.08	-6.87	-6.96	-0.19	0.06	0.10	0.08	-6.87	-6.96	-0.19	0.06	0.10	0.08
	TMLE	9.57	7.79	-0.11	0.37	0.18	0.10	11.81	8.76	-0.11	0.48	0.21	0.11	2.90	2.79	-0.05	0.12	0.06	0.04	2.90	2.79	-0.05	0.12	0.06	0.04
	TMLEc	10.84	10.82	-0.04	0.26	0.14	0.11	10.85	10.83	-0.04	0.26	0.14	0.11	3.10	3.07	-0.05	0.11	0.06	0.04	3.10	3.07	-0.05	0.11	0.06	0.04
	TMLEs	9.03	7.54	-0.11	0.35	0.17	0.10	9.34	7.69	-0.11	0.36	0.17	0.10	2.90	2.79	-0.05	0.12	0.06	0.04	2.90	2.79	-0.05	0.12	0.06	0.04
	TMLEr	9.04	7.54	-0.11	0.35	0.17	0.10	9.66	7.76	-0.11	0.39	0.18	0.10	2.90	2.79	-0.05	0.12	0.06	0.04	2.90	2.79	-0.05	0.12	0.06	0.04
ϕ_{12}	AB-GMM	-0.74	-0.69	-0.29	0.28	0.17	0.11	0.57	0.16	-0.60	0.63	0.39	0.23	-0.19	-0.16	-0.15	0.14	0.09	0.06	0.50	0.69	-0.21	0.22	0.13	0.09
	Sys-GMM	-0.73	-0.77	-0.22	0.20	0.13	0.08	-6.19	-6.11	-0.43	0.30	0.23	0.16	-0.70	-0.70	-0.12	0.10	0.07	0.04	-5.64	-5.63	-0.25	0.14	0.13	0.09
	FDLS	4.15	4.13	-0.16	0.24	0.13	0.09	4.14	4.12	-0.16	0.24	0.13	0.09	8.38	8.48	-0.06	0.22	0.12	0.09	8.38	8.48	-0.06	0.22	0.12	0.09
	TMLE	0.84	0.98	-0.20	0.21	0.13	0.08	-0.19	0.68	-0.27	0.22	0.15	0.08	-0.16	-0.12	-0.08	0.07	0.05	0.03	-0.16	-0.12	-0.08	0.07	0.05	0.03
	TMLEc	-8.84	-8.82	-0.22	0.04	0.12	0.09	-8.85	-8.83	-0.22	0.04	0.12	0.09	-3.75	-3.73	-0.11	0.03	0.06	0.04	-3.75	-3.73	-0.11	0.03	0.06	0.04
	TMLEs	0.98	0.97	-0.18	0.20	0.12	0.08	0.75	0.81	-0.19	0.20	0.12	0.08	-0.16	-0.12	-0.08	0.07	0.05	0.03	-0.16	-0.12	-0.08	0.07	0.05	0.03
	TMLEr	1.00	0.98	-0.18	0.20	0.12	0.08	0.51	0.63	-0.19	0.20	0.12	0.08	-0.16	-0.12	-0.08	0.07	0.05	0.03	-0.16	-0.12	-0.08	0.07	0.05	0.03

Table B.7: Design 1. Rejection frequencies for two sided t- tests for ϕ_{11}. True value $\phi_{11} = 0.6$.

	$N=100$ $T=3$ $\pi=1$					$N=100$ $T=3$ $\pi=3$					$N=100$ $T=6$ $\pi=1$					$N=100$ $T=6$ $\pi=3$				
	0.4	0.5	**0.6**	0.7	0.8	0.4	0.5	**0.6**	0.7	0.8	0.4	0.5	**0.6**	0.7	0.8	0.4	0.5	**0.6**	0.7	0.8
TMLE(r)	.320	.235	.210	.233	.316	.376	.291	.255	.259	.309	.802	.293	.106	.305	.664	.821	.346	.130	.270	.592
TMLEc(r)	.395	.139	.065	.172	.440	.399	.143	.072	.178	.444	.930	.414	.061	.465	.938	.929	.414	.061	.465	.938
TMLEs(r)	.257	.180	.172	.222	.340	.257	.181	.173	.224	.341	.780	.258	.092	.316	.695	.779	.255	.093	.318	.697
TMLEr(r)	.276	.201	.192	.236	.347	.314	.243	.233	.271	.363	.780	.258	.092	.316	.695	.779	.255	.093	.318	.697
AB-GMM2(W)	.039	.052	.088	.144	.225	.059	.087	.127	.176	.236	.111	.050	.154	.414	.725	.049	.081	.219	.450	.693
Sys-GMM2(W)	.399	.203	.087	.090	.231	.627	.491	.332	.176	.084	.931	.658	.227	.099	.462	.988	.957	.865	.646	.309
	$N=250$ $T=3$ $\pi=1$					$N=250$ $T=3$ $\pi=3$					$N=250$ $T=6$ $\pi=1$					$N=250$ $T=6$ $\pi=3$				
	0.4	0.5	**0.6**	0.7	0.8	0.4	0.5	**0.6**	0.7	0.8	0.4	0.5	**0.6**	0.7	0.8	0.4	0.5	**0.6**	0.7	0.8
TMLE(r)	.366	.198	.150	.234	.418	.407	.235	.177	.233	.386	.992	.540	.099	.510	.881	.994	.589	.134	.468	.827
TMLEc(r)	.760	.261	.056	.301	.780	.761	.260	.056	.300	.780		.789	.054	.812			.789	.054	.812	
TMLEs(r)	.330	.170	.133	.242	.449	.330	.170	.134	.242	.449	.987	.520	.085	.516	.894	.987	.520	.085	.516	.894
TMLEr(r)	.333	.173	.136	.245	.451	.343	.185	.150	.256	.456	.987	.520	.085	.516	.894	.987	.520	.085	.516	.894
AB-GMM2(W)	.060	.032	.066	.145	.275	.029	.048	.086	.137	.209	.406	.088	.094	.437	.840	.125	.047	.126	.377	.691
Sys-GMM2(W)	.585	.257	.076	.164	.524	.587	.397	.208	.094	.123	.992	.689	.092	.361	.947	.986	.881	.599	.253	.189

Table B.8: Design 2. Rejection frequencies for two sided t- tests for ϕ_{11}. True value $\phi_{11} = 0.4$.

	$N=100$ $T=3$ $\pi=1$					$N=100$ $T=3$ $\pi=3$					$N=100$ $T=6$ $\pi=1$					$N=100$ $T=6$ $\pi=3$				
	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6
TMLE(r)	.239	.185	.173	.205	.281	.315	.260	.238	.247	.292	.712	.219	.055	.273	.687	.704	.222	.065	.279	.681
TMLEc(r)	.358	.148	.064	.104	.272	.361	.151	.069	.108	.276	.808	.311	.050	.258	.756	.808	.311	.050	.258	.756
TMLEs(r)	.204	.153	.148	.193	.284	.218	.166	.159	.201	.288	.713	.219	.054	.272	.688	.713	.219	.054	.272	.688
TMLEr(r)	.211	.160	.155	.198	.287	.250	.200	.195	.232	.311	.713	.219	.054	.272	.688	.713	.219	.055	.273	.688
AB-GMM2(W)	.057	.046	.067	.123	.211	.038	.053	.076	.114	.165	.183	.062	.094	.274	.565	.090	.049	.100	.246	.462
Sys-GMM2(W)	.278	.145	.081	.086	.151	.531	.424	.311	.208	.135	.835	.528	.209	.077	.200	.971	.925	.838	.693	.494
	$N=250$ $T=3$ $\pi=1$					$N=250$ $T=3$ $\pi=3$					$N=250$ $T=6$ $\pi=1$					$N=250$ $T=6$ $\pi=3$				
	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6
TMLE(r)	.297	.116	.103	.215	.409	.336	.170	.149	.240	.399	.985	.498	.048	.513	.963	.985	.498	.048	.513	.962
TMLEc(r)	.716	.280	.058	.150	.515	.716	.280	.058	.150	.515	.997	.644	.043	.522	.989	.997	.644	.043	.522	.989
TMLEs(r)	.285	.102	.091	.209	.414	.290	.106	.095	.213	.416	.985	.498	.048	.513	.963	.985	.498	.048	.513	.963
TMLEr(r)	.286	.102	.092	.210	.414	.295	.113	.104	.221	.422	.985	.498	.048	.513	.963	.985	.498	.048	.513	.963
AB-GMM2(W)	.128	.051	.059	.145	.295	.035	.034	.058	.106	.179	.517	.138	.068	.356	.775	.267	.077	.074	.268	.591
Sys-GMM2(W)	.434	.171	.067	.129	.350	.465	.318	.197	.120	.101	.949	.536	.086	.226	.765	.950	.811	.573	.319	.167

Table B.9: Design 3. Rejection frequencies for two sided t- tests for ϕ_{11}. True value $\phi_{11} = 0.4$.

	$N=100$ $T=3$ $\pi=1$					$N=100$ $T=3$ $\pi=3$					$N=100$ $T=6$ $\pi=1$					$N=100$ $T=6$ $\pi=3$				
	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6
TMLE(r)	.303	.222	.210	.249	.329	.733	.331	.141	.358	.701	.784	.269	.061	.318	.766	.970	.562	.079	.590	.964
TMLEc(r)	.634	.393	.192	.089	.081	.980	.969	.950	.908	.833	.868	.450	.082	.056	.299	.734	.645	.529	.407	.289
TMLEs(r)	.218	.109	.108	.185	.335	.738	.289	.081	.328	.720	.789	.269	.057	.315	.769	.972	.562	.076	.588	.966
TMLEr(r)	.233	.130	.132	.210	.357	.739	.290	.083	.330	.722	.789	.269	.057	.315	.769	.972	.562	.076	.588	.966
AB-GMM2(W)	.039	.050	.070	.103	.152	.332	.133	.069	.193	.432	.087	.052	.103	.254	.477	.426	.151	.084	.370	.715
Sys-GMM2(W)	.872	.764	.616	.422	.232					.999	.996	.970	.855	.613	.313					
	$N=250$ $T=3$ $\pi=1$					$N=250$ $T=3$ $\pi=3$					$N=250$ $T=6$ $\pi=1$					$N=250$ $T=6$ $\pi=3$				
	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6
TMLE(r)	.486	.180	.135	.259	.480	.982	.579	.079	.580	.962	.995	.577	.049	.595	.985		.910	.060	.907	
TMLEc(r)	.961	.771	.392	.112	.067			.999	.998	.987	.999	.885	.194	.055	.532	.901	.870	.812	.694	.517
TMLEs(r)	.512	.110	.060	.233	.536	.984	.575	.067	.576	.966	.995	.577	.049	.595	.985		.910	.060	.907	
TMLEr(r)	.514	.114	.065	.239	.541	.984	.575	.067	.576	.966	.995	.577	.049	.595	.985		.910	.060	.907	
AB-GMM2(W)	.030	.030	.047	.089	.149	.697	.270	.063	.339	.748	.259	.067	.080	.294	.618	.782	.326	.065	.509	.910
Sys-GMM2(W)	.995	.972	.898	.719	.389							.998	.939	.674	.295					

Table B.10: Design 4. Rejection frequencies for two sided t- tests for ϕ_{11}. True value $\phi_{11} = 0.4$.

	$N=100$ $T=3$ $\pi=1$					$N=100$ $T=3$ $\pi=3$					$N=100$ $T=6$ $\pi=1$					$N=100$ $T=6$ $\pi=3$				
	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6
TMLE(r)	.260	.186	.179	.225	.309	.553	.256	.149	.291	.549	.776	.265	.064	.307	.741	.951	.522	.075	.526	.941
TMLEc(r)	.570	.314	.141	.065	.093	.922	.875	.798	.693	.561	.873	.440	.080	.069	.342	.680	.540	.380	.251	.152
TMLEs(r)	.198	.116	.119	.186	.311	.594	.209	.071	.246	.583	.784	.266	.058	.303	.745	.958	.526	.072	.526	.948
TMLEr(r)	.207	.126	.129	.195	.318	.595	.214	.078	.254	.590	.784	.266	.058	.303	.745	.958	.526	.072	.526	.948
AB-GMM2(W)	.165	.075	.071	.141	.283	.488	.186	.082	.233	.535	.399	.114	.084	.319	.696	.813	.326	.086	.498	.906
Sys-GMM2(W)	.697	.569	.445	.329	.238					.999	.977	.909	.779	.615	.475					
	$N=250$ $T=3$ $\pi=1$					$N=250$ $T=3$ $\pi=3$					$N=250$ $T=6$ $\pi=1$					$N=250$ $T=6$ $\pi=3$				
	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6
TMLE(r)	.434	.169	.136	.257	.458	.878	.415	.090	.439	.859	.993	.569	.050	.574	.982		.870	.059	.861	
TMLEc(r)	.930	.666	.259	.059	.109	.996	.995	.987	.957	.861		.880	.179	.074	.605	.901	.851	.719	.488	.268
TMLEs(r)	.440	.098	.066	.222	.485	.934	.423	.061	.437	.900	.993	.569	.049	.574	.982		.870	.059	.861	
TMLEr(r)	.443	.101	.069	.224	.487	.934	.423	.061	.438	.900	.993	.569	.049	.574	.982		.870	.059	.861	
AB-GMM2(W)	.390	.126	.060	.194	.478	.845	.357	.070	.390	.849	.820	.282	.062	.465	.921	.993	.647	.064	.743	.996
Sys-GMM2(W)	.917	.804	.663	.517	.384						.999	.985	.900	.728	.559					

Table B.11: Design 5. Rejection frequencies for two sided t- tests for ϕ_{11}. True value $\phi_{11} = 0.4$.

	$N=100$ $T=3$ $\pi=1$					$N=100$ $T=3$ $\pi=3$					$N=100$ $T=6$ $\pi=1$					$N=100$ $T=6$ $\pi=3$				
	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6
TMLE(r)	.280	.242	.230	.246	.282	.357	.316	.293	.289	.299	.429	.141	.090	.234	.494	.433	.172	.123	.254	.490
TMLEc(r)	.232	.116	.079	.116	.224	.241	.127	.092	.128	.235	.537	.185	.066	.202	.525	.536	.185	.066	.203	.525
TMLEs(r)	.226	.190	.188	.221	.278	.245	.207	.202	.230	.282	.429	.134	.079	.228	.495	.431	.137	.082	.230	.497
TMLEr(r)	.240	.206	.203	.234	.288	.305	.272	.266	.285	.323	.429	.134	.080	.229	.496	.434	.141	.089	.237	.502
AB-GMM2(W)	.054	.061	.087	.133	.198	.053	.071	.098	.132	.175	.090	.065	.144	.322	.547	.069	.072	.148	.295	.481
Sys-GMM2(W)	.268	.165	.101	.080	.106	.579	.485	.389	.285	.189	.823	.629	.389	.191	.117	.986	.971	.939	.880	.765
	$N=250$ $T=3$ $\pi=1$					$N=250$ $T=3$ $\pi=3$					$N=250$ $T=6$ $\pi=1$					$N=250$ $T=6$ $\pi=3$				
	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6
TMLE(r)	.234	.162	.155	.218	.328	.303	.232	.214	.255	.329	.826	.279	.059	.345	.785	.823	.280	.064	.348	.782
TMLEc(r)	.443	.169	.063	.124	.352	.443	.169	.063	.125	.352	.896	.386	.051	.336	.848	.896	.386	.051	.336	.848
TMLEs(r)	.208	.136	.134	.206	.333	.219	.147	.143	.214	.337	.827	.279	.058	.346	.785	.826	.279	.058	.346	.785
TMLEr(r)	.211	.140	.137	.209	.336	.239	.168	.166	.236	.354	.827	.279	.058	.346	.785	.827	.279	.059	.346	.785
AB-GMM2(W)	.072	.048	.067	.138	.237	.033	.045	.072	.110	.161	.254	.069	.082	.293	.628	.136	.054	.087	.239	.493
Sys-GMM2(W)	.316	.154	.078	.093	.206	.504	.391	.278	.182	.115	.882	.551	.185	.084	.297	.972	.932	.844	.701	.501

Table B.12: Design 6. Rejection frequencies for two sided t- tests for ϕ_{11}. True value $\phi_{11} = 0.4$.

	$N=100$ $T=3$ $\pi=1$					$N=100$ $T=3$ $\pi=3$					$N=100$ $T=6$ $\pi=1$					$N=100$ $T=6$ $\pi=3$				
	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6
TMLE(r)	.326	.223	.171	.168	.204	.379	.283	.227	.211	.228	.804	.324	.059	.170	.550	.802	.324	.061	.172	.549
TMLEc(r)	.560	.297	.126	.065	.119	.561	.301	.132	.072	.127	.864	.407	.070	.168	.615	.864	.407	.070	.168	.615
TMLEs(r)	.301	.194	.146	.151	.199	.316	.208	.159	.161	.205	.804	.324	.059	.170	.550	.804	.324	.059	.170	.550
TMLEr(r)	.305	.201	.154	.157	.202	.336	.236	.191	.191	.229	.804	.324	.059	.170	.550	.804	.324	.059	.170	.550
AB-GMM2(W)	.063	.048	.070	.123	.213	.045	.057	.080	.116	.163	.156	.057	.099	.274	.548	.069	.050	.108	.243	.440
Sys-GMM2(W)	.288	.157	.085	.082	.136	.582	.482	.375	.267	.174	.830	.531	.214	.079	.191	.972	.927	.844	.696	.501
	$N=250$ $T=3$ $\pi=1$					$N=250$ $T=3$ $\pi=3$					$N=250$ $T=6$ $\pi=1$					$N=250$ $T=6$ $\pi=3$				
	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6	0.2	0.3	**0.4**	0.5	0.6
TMLE(r)	.560	.231	.110	.110	.239	.572	.263	.144	.138	.248	.996	.701	.073	.294	.883	.996	.701	.073	.294	.883
TMLEc(r)	.918	.617	.217	.054	.181	.918	.618	.217	.054	.181	.999	.786	.092	.303	.942	.999	.786	.092	.303	.942
TMLEs(r)	.557	.222	.102	.105	.239	.561	.226	.106	.108	.241	.996	.701	.073	.294	.883	.996	.701	.073	.294	.883
TMLEr(r)	.557	.222	.102	.105	.240	.563	.231	.114	.118	.249	.996	.701	.073	.294	.883	.996	.701	.073	.294	.883
AB-GMM2(W)	.142	.055	.058	.143	.299	.033	.036	.059	.103	.166	.467	.120	.071	.342	.747	.207	.064	.076	.254	.540
Sys-GMM2(W)	.437	.171	.068	.120	.313	.508	.370	.256	.168	.122	.954	.543	.087	.223	.758	.952	.819	.573	.317	.170